Operations research is a field that is rich in applied mathematical problems from business and economics. Since problems in this field can be computationally difficult and time consuming to solve software packages such as Lindo are often times used. However, typical software packages are not able to give students insight to the mathematical techniques used to solve linear programming problems since only the problem solutions are output. Thus, these programs become little more than black-boxes to students. In this paper we will demonstrate several Visual Basic programs linked to Excel that students can use as an aid to solve such problems yet also require students to demonstrate an understanding of the solution techniques employed, thus, bridging the gap between problem setups and solution output.

We begin with a typical linear programming problem.

Example: A small clothing manufacturer produces shirts, pants and coats. Each shirt can be sold for $60, each pants for $40 and each coat for $50. Production of each requires labor, such as, cutting minutes, sewing minutes and ironing minutes according to Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Shirt</th>
<th>Pants</th>
<th>Coat</th>
<th>Resource Limitation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting</td>
<td>10</td>
<td>8</td>
<td>2</td>
<td>50</td>
</tr>
<tr>
<td>Sewing</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>Ironing</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 1 Clothing Resource Requirements

Also, the maximum number of shirts that can be sold is 10 and the maximum number of pants that can be sold in 6. The problem can then be summarized as follows,

maximize $z = 60x_1 + 40x_2 + 50x_3$
subject to: $10x_1 + 8x_2 + 2x_3 \leq 50$
$5x_1 + 3x_2 + 4x_3 \leq 9$
$5x_1 + 5x_2 + 2x_3 \leq 10$
$x_1 \leq 10$
$x_2 \leq 6$
$x_1, x_2, x_3 \geq 0$
The problem is placed in standard form whereby slack variables are added.

\[
\begin{align*}
\text{maximize } & \quad z = 60x_1 + 50x_2 + 40x_3 + 0s_1 + 0s_2 + 0s_3 \\
\text{subject to: } & \quad 10x_1 + 8x_2 + 2x_3 + s_1 = 50 \\
& \quad 5x_1 + 3x_2 + 4x_3 + s_2 = 9 \\
& \quad 5x_1 + 5x_2 + 2x_3 + s_3 = 10 \\
& \quad x_2 + s_4 = 10 \\
& \quad x_3 + s_5 = 6 \\
& \quad x_1, x_2, x_3, s_1, s_2, s_3, s_4, s_5 \geq 0
\end{align*}
\]

The simplex method is performed by assuming an initial non-optimized solution \( s_1 = 50, s_2 = 9, s_3 = 10, s_4 = 10, s_5 = 6 \) and \( x_1 = x_2 = x_3 = 0 \). The non-zero set of variables \( \{s_1, s_2, s_3, s_4, s_5\} \) is called the basic variable set (BV) and the set of zero variables \( \{x_1, x_2, x_3\} \) is called the non-basic variable set (NBV).

When the simplex method is performed a variable in the basic set is chosen and increased to a maximum allowed value thus entering the BV set. In order for a non-basic variable to be increased and keep the constraint equations true the basic variables will change and one or more of them will decrease to 0 and enter the NBV set. For example, if we choose to increase \( x_1 \) the largest it can be increased to is \( \min\{\frac{50}{10}, \frac{9}{3}, \frac{10}{5}, \infty, \infty\} \). If \( x_1 \) were increased more than this than one of the basic variables would have to become negative which violates the constraints. Thus, we see that if we set \( x_1 = \frac{9}{3} \) then \( s_2 \) must decrease to 0 and enter NBV, and \( s_1 \) and \( s_3 \) must decrease to 30 and 1 respectively. The objective function \( z = 60x_1 + 50x_2 + 40x_3 \) then increases from \( z = 0 \) under the initial non-optimized solution of

\[
\text{BV} = \{s_1 = 50, s_2 = 9, s_3 = 10, s_4 = 10, s_5 = 6\} \\
\text{NBV} = \{x_1 = 0, x_2 = 0, x_3 = 0\}
\]

to \( z = 60 \cdot \frac{9}{3} = 108 \) under the still non-optimized but larger \( z \) – value solution of

\[
\text{BV} = \{x_1 = 9/5, s_1 = 30, s_3 = 1, s_4 = 10, s_5 = 6\} \\
\text{NBV} = \{s_2 = 0, x_2 = 0, x_3 = 0\}.
\]

Notice how \( x_1 \) moved from the NBV set to the BV set which then forced \( s_2 \) to move from the BV set to the NBV set and during this process \( z \) increased from 0 to 108.

The process of moving a variable from the NBV set to the BV set so that the objective function value is increased can be quickly performed by a pivot of the following matrix representation of the standard form problem as shown below.
\begin{align*}
x_1 & \quad x_2 & \quad x_3 & \quad s_1 & \quad s_2 & \quad s_3 & \quad s_4 & \quad s_5 & \quad z \\
-60 & \quad -40 & \quad -50 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 0 \\
10 & \quad 8 & \quad 2 & \quad 1 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 50 \\
5 & \quad 3 & \quad 4 & \quad 0 & \quad 1 & \quad 0 & \quad 0 & \quad 0 & \quad 9 \\
5 & \quad 5 & \quad 2 & \quad 0 & \quad 0 & \quad 1 & \quad 0 & \quad 0 & \quad 10 \\
0 & \quad 1 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 1 & \quad 0 & \quad 10 \\
0 & \quad 0 & \quad 1 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 1 & \quad 6 \\
\end{align*}

\[ BV = \{ x_1, s_1, s_3, s_4, s_5 \} \]
\[ NBV = \{ x_1, x_2, x_3 \} \]

Pivoting about the row 2 column 1 entry of 5 where column 1 corresponds the variable chosen to enter the basis we get,

\begin{align*}
x_1 & \quad x_2 & \quad x_3 & \quad s_1 & \quad s_2 & \quad s_3 & \quad s_4 & \quad s_5 & \quad z \\
0 & \quad -4 & \quad -2 & \quad 0 & \quad 12 & \quad 0 & \quad 0 & \quad 0 & \quad 108 \\
0 & \quad 2 & \quad 0 & \quad 1 & \quad -2 & \quad 0 & \quad 0 & \quad 0 & \quad 32 \\
1 & \quad 0.6 & \quad 0.8 & \quad 0 & \quad 0.2 & \quad 0 & \quad 0 & \quad 0 & \quad 1.8 \\
0 & \quad 2 & \quad -2 & \quad 0 & \quad -1 & \quad 1 & \quad 0 & \quad 0 & \quad 1 \\
0 & \quad 1 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 1 & \quad 0 & \quad 10 \\
0 & \quad 0 & \quad 1 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 1 & \quad 6 \\
\end{align*}

\[ BV = \{ x_1 = 1.8, s_1 = 32, s_3 = 1, s_4 = 10, s_5 = 6 \} \]
\[ NBV = \{ x_2 = 0, x_3 = 0, s_2 = 0 \} \]

Considering the new objective function row,

\[
(0 \quad -4 \quad -2 \quad 0 \quad 12 \quad 0 \quad 0 \quad 0 \quad 108)
\]

notice that the new objective function value of 108 appears in the z-column. Using this row we can decide if another variable can be moved from NBV to BV and increase z further. The value -4 indicates that for every unit increase in \( x_2 \) z will further increase by 4 and the value of -2 indicates that for every unit increase in \( x_3 \) z will further increase by 2. The value of 12 indicates that if \( s_2 \) is increased by one unit z will decrease by 12 so we do not want to select \( s_2 \) to enter the basis. Since by increasing \( x_2 \) z will increase at the fastest rate we choose \( x_2 \) to become a basic variable. \( x_2 \) can be increased up to \( \min \{ \frac{32}{2}, \frac{1.8}{0.6}, \frac{2}{1}, \frac{10}{1} \} = 2 \). Pivoting about the row 4, column 2 entry of 2 yields the new matrix,

\begin{align*}
x_1 & \quad x_2 & \quad x_3 & \quad s_1 & \quad s_2 & \quad s_3 & \quad s_4 & \quad s_5 & \quad z \\
0 & \quad 0 & \quad -6 & \quad 0 & \quad 10 & \quad 2 & \quad 0 & \quad 0 & \quad 110 \\
0 & \quad 0 & \quad 2 & \quad 1 & \quad -1 & \quad -1 & \quad 0 & \quad 0 & \quad 31 \\
1 & \quad 0 & \quad 1.4 & \quad 0 & \quad 0.5 & \quad -0.3 & \quad 0 & \quad 0 & \quad 1.5 \\
0 & \quad 1 & \quad -1 & \quad 0 & \quad -0.5 & \quad 0.5 & \quad 0 & \quad 0 & \quad 0.5 \\
\end{align*}
By the same logic, we see that we can the objective function value $z$ which is currently
110 can be increased by moving $x_3$ from the NBV set to the BV set. We can increase $x_3$
up to the maximum value of $\min\{\frac{31}{2}, \frac{15}{4}, \infty, \frac{9.5}{1}, \frac{6}{1}\} = \frac{15}{14}$. Notice if $x_3$ is to be
increased then row 4,

$$
\begin{pmatrix}
0 & 1 & -1 & 0 & -0.5 & 0.5 & 1 & 0 & 0.5 \\
\end{pmatrix}
$$

corresponds to the equation $x_2 = 0.5 + x_3$ so $x_3$ can be increased without bound without
forcing $x_2$ to leave BV. Thus the $\infty$ in $\min\{\frac{31}{2}, \frac{15}{4}, \infty, \frac{9.5}{1}, \frac{6}{1}\}$. Pivoting about the value
1.4, then yields,

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.29</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12.14</td>
<td>0.71</td>
<td>0</td>
<td>0</td>
<td>116.43</td>
</tr>
<tr>
<td>-1.43</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1.71</td>
<td>-0.57</td>
<td>0</td>
<td>0</td>
<td>28.86</td>
</tr>
<tr>
<td>0.71</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.36</td>
<td>-0.21</td>
<td>0</td>
<td>0</td>
<td>1.07</td>
</tr>
<tr>
<td>0.71</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-0.14</td>
<td>0.29</td>
<td>0</td>
<td>0</td>
<td>1.57</td>
</tr>
<tr>
<td>-0.71</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.14</td>
<td>-0.29</td>
<td>1</td>
<td>0</td>
<td>8.43</td>
</tr>
<tr>
<td>-0.71</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.36</td>
<td>0.21</td>
<td>0</td>
<td>1</td>
<td>4.93</td>
</tr>
</tbody>
</table>

$BV = \{x_2 = 1.57, x_3 = 1.07, s_1 = 28.86, s_4 = 8.43, s_5 = 4.93\}$
$NBV = \{x_1 = 0, s_2 = 0, s_3 = 0\}$

At this point we see that the objective function row has no negative values in it which
indicates that no variable can be moved from NBV to BV and increase $z$. Therefore, an
optimal solution has been found, namely, $x_1 = 0, x_2 = 1.57, x_3 = 1.07$ yielding an
optimal objective function value $z = 116.43$.

For the pivoting process in this example we use the following VBA Excel macro. The
VBA editor can be accessed from Excel by selecting View → Macros. If no macro
currently exists a new one must be created and then select Edit to open the VBA editor.

Sub Simplex_Pivot()
' Macro recorded 5/27/2012 by David Nawrocki
' Keyboard Shortcut: Ctrl+p
    Dim RowDim As Integer
    Dim ColDim As Integer
    RowDim = 0
    ColDim = 0
Do
  RowDim = RowDim + 1
Loop While Cells(RowDim + 1, 1) <> ""
Do
  ColDim = ColDim + 1
Loop While Cells(1, ColDim + 1) <> ""
R = ActiveCell.Row
C = ActiveCell.Column
For Row = 0 To RowDim - 1
  P1 = Cells(Row + 1, C).Value
  P2 = Cells(R, C).Value
  For Column = 0 To ColDim - 1
    If Row + 1 = R Then
      Cells(Row + 1, Column + 1).Value = Cells(Row + 1, Column + 1).Value / P2
    Else
      Cells(Row + 1, Column + 1).Value = Cells(Row + 1, Column + 1).Value -
        Cells(R, Column + 1).Value * P1 / P2
    End If
  Next Column
Next Row
End Sub

The Keyboard Shortcut can be set as a macro option in the Macro dialog box. For this example, the macro is executed from the worksheet anytime the keystrokes CTRL+p are typed.

**Unbounded Linear Programming Problems**

Our macro can also be used to easily identify various problems that can arise with linear programming problems such as cycling, infeasible problems and unbounded problems. We will now demonstrate how to identify unbounded linear programming problems. Consider the following linear program,

\[
\begin{align*}
\text{maximize } z & = 36x_1 + 30x_2 + 3x_3 + 4x_4 \\
\text{subject to: } x_1 + x_2 - x_3 & \leq 5 \\
6x_1 + 5x_2 - 4x_4 & \leq 9 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
\]

To solve this problem we set up the following pivoting table.

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-36</td>
<td>-30</td>
<td>-3</td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>9</td>
</tr>
</tbody>
</table>
By performing the same pivoting technique as described in the previous example we arrive at the following table.

\[
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \\
0 & 7 & -45 & 0 & 42 & -1 & 200 \\
0 & 0.25 & -1.5 & 1 & 1.5 & -0.25 & 5 \\
1 & 1 & -1 & 0 & 1 & 0 & 5 \\
\end{array}
\]

At this point the basic and non-basic variable sets are as follows,

\[BV = \{x_1 = 5, x_4 = 5\}\]
\[NBV = \{x_2 = 0, x_3 = 0, s_1 = 0, s_2 = 0\}.\]

We see that for each unit increase in \(x_3\), \(z\) will increase by 45. Furthermore, since all the coefficients in the \(x_3\) column are negative, \(x_3\) can be increased without bound. That is, as \(x_3\) is increased no other variables from the basic variable set will be forced to zero. Thus, the problem is unbounded.

**Solving Transportation Problems**

We can also use this macro to illustrate solution techniques for transportation problems.

Example: ElectricCo has three electric power plants that supply electricity for four cities. Each power plant has the following capacities: plant 1 can supply 45 mwh, plant 2 can supply 62 mwh and plant 3 can supply 50 mwh. Assume the peak power demands in these cities occur at the same time and are: city 1 requires 55 mwh, city 2 requires 34 mwh, city 3 requires 28 mwh and city 4 requires 40 mwh. The costs of sending 1 mwh of energy from plant \(i\) to city \(j\) is summarized in Table 2 below.

<table>
<thead>
<tr>
<th></th>
<th>City 1</th>
<th>City 2</th>
<th>City 3</th>
<th>City 4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant 1</td>
<td>8</td>
<td>6</td>
<td>9</td>
<td>10</td>
<td>45</td>
</tr>
<tr>
<td>Plant 2</td>
<td>9</td>
<td>10</td>
<td>14</td>
<td>6</td>
<td>62</td>
</tr>
<tr>
<td>Plant 3</td>
<td>14</td>
<td>9</td>
<td>15</td>
<td>7</td>
<td>50</td>
</tr>
<tr>
<td>Demand</td>
<td>55</td>
<td>34</td>
<td>28</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>

Table 2 ElectricCo Costs

Let \(c_{ij}\) be cost of sending 1 mwh of electricity from plant \(i\) to city \(j\) and \(x_{ij}\) be number of mwh shipped from plant \(i\) to city \(j\). Then the linear program may be written as:
minimize $z = \sum_{i,j} c_{ij} \cdot x_{ij}$

s.t. \[ \sum_j x_{ij} \leq S_i \]
\[ \sum_i x_{ij} \geq D_j \]

For this particular problem,

\begin{align*}
\text{minimize } z &= 8x_{11} + 6x_{12} + 9x_{13} + 10x_{14} + 9x_{21} + 10x_{22} + 14x_{23} + 6x_{24} + \\
& \quad 14x_{31} + 9x_{32} + 15x_{33} + 6x_{34} \\
\text{s.t. } x_{11} + x_{12} + x_{13} + x_{14} &\leq 45 \\
x_{21} + x_{22} + x_{23} + x_{24} &\leq 62 \\
x_{31} + x_{32} + x_{33} + x_{34} &\leq 50 \\
x_{11} + x_{21} + x_{31} &\geq 55 \\
x_{12} + x_{22} + x_{32} &\geq 34 \\
x_{13} + x_{23} + x_{33} &\geq 28 \\
x_{14} + x_{24} + x_{34} &\geq 40
\end{align*}

Notice for this problem that the total supply and demand are balanced. That is to say, \( \sum S_i = \sum D_i \). If this were not the case then we would have to balance the problem by adding a dummy demand point if total supply exceeded total demand or by adding an additional dummy supply point and perhaps a penalty to the objective function if total demand exceeded total supply.

Before we can use the pivoting program to solve this problem we must find a Basic Feasible. Several common methods to do this are the Northwest corner method, the minimum cost method and Vogel's method. We will use the Northwest corner method here since it is the easiest although the other methods will in general yield initial basic feasible solutions that are closer to the optimal solution. Applying the Northwest corner method yields Table 3 below.

<table>
<thead>
<tr>
<th></th>
<th>City 1</th>
<th>City 2</th>
<th>City 3</th>
<th>City 4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant 1</td>
<td>45</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>Plant 2</td>
<td>10</td>
<td>34</td>
<td>18</td>
<td>0</td>
<td>62</td>
</tr>
<tr>
<td>Plant 3</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>Demand</td>
<td>55</td>
<td>24</td>
<td>28</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 ElectricCo b.f.s.

This gives us a starting basic feasible solution with basic variable set given by,

\[ \text{BV} = \{ x_{11} = 45, x_{21} = 10, x_{22} = 34, x_{23} = 18, x_{33} = 10, x_{34} = 40 \} \]
We can use this information to begin a pivoting process to solve the problem. Starting with $x_{11}$, we select a row to pivot about by choosing the minimum of the values in the rightmost column. Notice we can pivot about the row corresponding to 45 or that corresponding to the value 55 in the rightmost column. Since Plant 1 has at most 45 mwh of output, we must choose to pivot about the row corresponding to 45 (row 2).

$$
\begin{array}{cccccccccccc}
\text{x11} & \text{x12} & \text{x13} & \text{x14} & \text{x21} & \text{x22} & \text{x23} & \text{x24} & \text{x31} & \text{x32} & \text{x33} & \text{x34} \\
-8 & -6 & -9 & -10 & -9 & -14 & -6 & -14 & -9 & -15 & -6 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 62 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 55 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 34 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 28 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 40 \\
\end{array}
$$

By pivoting about the other basic variable columns in a similar manner we obtain the following table.

$$
\begin{array}{cccccccccccc}
\text{x11} & \text{x12} & \text{x13} & \text{x14} & \text{x21} & \text{x22} & \text{x23} & \text{x24} & \text{x31} & \text{x32} & \text{x33} & \text{x34} \\
0 & 3 & 4 & -6 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 1432 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 10 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 34 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 10 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 40 \\
\end{array}
$$

From here we see that by increasing $x_{13}$ we can further reduce the current cost of 1432. Doing so will yield the next table.

$$
\begin{array}{cccccccccccc}
\text{x11} & \text{x12} & \text{x13} & \text{x14} & \text{x21} & \text{x22} & \text{x23} & \text{x24} & \text{x31} & \text{x32} & \text{x33} & \text{x34} \\
0 & 3 & 0 & -10 & 0 & 0 & 0 & -1 & 0 & 0 & 1360 \\
1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 27 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 28 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 34 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 10 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 40 \\
\end{array}
$$
Notice that $x_{13}$ has entered the basis and $x_{23}$ has left the basis. The basic variable set is now,

$$BV = \{x_{11} = 27, x_{13} = 18, x_{21} = 28, x_{22} = 34, x_{33} = 10, x_{34} = 40\}.$$ 

Continuing in this manner the optimal table is,

\[
\begin{array}{cccccccccc}
 & x_{11} & x_{12} & x_{13} & x_{14} & x_{21} & x_{22} & x_{23} & x_{24} & x_{31} & x_{32} & x_{33} & x_{34} \\
-2 & 0 & 0 & -7 & 0 & -1 & -2 & 0 & -5 & 0 & -3 & 0 & 1242 \\
1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 17 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 28 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 55 \\
-1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 7 \\
-1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 17 \\
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 33 \\
\end{array}
\]

with $BV = \{x_{12} = 17, x_{13} = 28, x_{21} = 55, x_{24} = 7, x_{32} = 17, x_{34} = 33\}$. Our optimal solution can now be summarized as shown in Table 4.

<table>
<thead>
<tr>
<th></th>
<th>City 1</th>
<th>City 2</th>
<th>City 3</th>
<th>City 4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant 1</td>
<td>0</td>
<td>17</td>
<td>28</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>Plant 2</td>
<td>55</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>62</td>
</tr>
<tr>
<td>Plant 3</td>
<td>0</td>
<td>17</td>
<td>0</td>
<td>33</td>
<td>50</td>
</tr>
<tr>
<td>Demand</td>
<td>55</td>
<td>24</td>
<td>28</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>

Table 4 Optimal ElectricCo Solution

REFERENCES