Abstract: The Mathematics Department at The College of Saint Rose adopted the Harvard Calculus program in the Fall of 1992. That same semester, we inaugurated our mathematics computer classroom. This room has twenty-one computers equipped with the Maple V computer algebra system. This talk will present several examples on the interactive use of this software to enhance and develop several topics from the CCH program.
I teach at the College of Saint Rose in Albany, NY and our Mathematics Department is fortunate enough to have its own computer classroom. The classroom has twenty-one computers so it can accommodate classes of up to twenty students. All of the computers have the Maple V computer algebra system on them and nearly all of our mathematics courses are taught in this room. So what I’d like to do today is give a few examples that show how I use this facility to present some of the topics in the Harvard Calculus program that we’ve been using for the past four years. But before I do, I want to give a brief description of how it’s laid out. See Figure 1 for a not-to-scale-diagram.

As you can see from this slide, we do not use desks in this room; rather, we use an array of tables laid out in a stylized “W”-pattern. The tables on the outer perimeter each hold six computers and the center arm of the “W” holds another two. The twenty-first computer is at the instructors desk at the front of the room. The tables forming the arms of the W serve as work tables for the students and the chairs where the students sit are located in the aisles between the work tables and the computer tables. The chairs are armless and cushioned; they are on rollers and swivel so that if a student is taking notes and needs to use a computer, he or she can just swivel 180 degrees and a computer will be right there. The computer at the front desk is connected to two large overhead monitors so that when the instructor wishes to demonstrate a point, it will be visible to the entire class. The room is large and airy and the aisles are wide enough so that the instructor can easily navigate them allowing him or her to interact with the students at the work tables in a variety of ways. Another advantage is that all the computer monitors face either to the side or front of the room making them easily visible to the instructor.

As I had mentioned before, I’d like to present a few examples that show how I use this facility to develop some of the topics in the Harvard Calculus program. Now I like to run the class in a sort of lecture/lab/recitation manner and generally speaking, I try to determine how well the topic lends itself to the type of exploratory/discovery activities I try to get my students involved with. Fortunately, most of the topics in Harvard Calculus program lend themselves to these kinds of activities rather well. For example, one of the topics I like to use it for is to help develop some of the formulas for finding derivatives of functions. For example, suppose that on a particular day, I want to develop the formula for the derivative of $f(x) = x^n$. 
These tables hold six computers.

Work tables for six students.

This is the instructor’s table; it has a computer and a printer.

This table holds two computers and also acts as a desk for two students one on either side.

Work tables for six students.

These tables hold six computers.

Figure 1. Layout of the Mathematics Computer Classroom
Now the students are already familiar with derivative having worked with it graphically and numerically in a previous chapter. They also are well-acquainted with the fact that the derivative is the limit of the difference quotient. So, after they have started Maple on their respective computers, I’ll ask them to either read in or define the following Maple procedure, calling it dq:

\[
\text{dq}(f,x,h) \rightarrow \frac{f(x+h) - f(x)}{h}
\]

Next, I’ll have them define a function whose derivative we seek, say \( f(x) = x^3 \).

\[
f := x \rightarrow x^3;
\]

Before proceeding to the determination of the derivative, we’ll plot the function together with its difference quotient for a small value of \( h \), say \( h = 0.1 \).

\[
>\text{plot( \{f(x), dq(f,x,.1)\}, x = -3 .. 3);}
\]
The plotting is followed by a brief discussion to determine which graph represents which function. Once the graph of difference quotient is determined, we follow with a discussion on the shape of its graph (parabolic) and what kinds of functions might have shapes like this one (quadratic functions). So, at this point the students have determined that the derivative of \( f(x) = x^3 \) is probably a quadratic function.

The difference quotient for this function is determined next; I emphasize here that we wish to consider the difference quotient as a function of \( h \) since the derivative is its limit as \( h \rightarrow 0 \).

\[
> dq(f, x, h);
\]

\[
\frac{(x+h)^3 - x^3}{h}
\]

The next step is to simplify this expression into a more useful form:

\[
> simplify(dq(f, x, h));
\]

\[
3x^2 + 3xh + h^2
\]
We’ll define the simplified expression as a function of \( h \), then evaluate it for some small values of \( h \). Computing \( g(h) \) for several small values of \( h \) gives the class a good idea of what may happen in the limit.

\[
g := h \mapsto 3x^2 + 3xh + h^2;
\]

\[
\text{for i to 5 do } g(.1^i) \text{ od};
\]

\[
\begin{align*}
3x^2 + .3x + .01 \\
3x^2 + .03x + .0001 \\
3x^2 + .003x + .1 \times 10^{-5} \\
3x^2 + .0003x + .1 \times 10^{-7} \\
3x^2 + .00003x + .1 \times 10^{-9}
\end{align*}
\]

At this point we have enough experimental evidence to conclude that the derivative of \( f(x) = x^3 \) is \( f'(x) = 3x^2 \). I’ll then suggest that we try the same procedures on another power function, \( f(x) = x^4 \).

\[
f := x \mapsto x^4;
\]

\[
\text{plot( } \{f(x), dq(f, x, .1)\}, x = -3 .. 3);\]
Fresh from their experience in the last exercise, the students readily recognize that one of these is the graph of \( f(x) = x^3 \) or at least some multiple of it; they also know that the same graph is the graph of the difference quotient for \( h = 0.1 \) since the other graph lies entirely above the \( x \)-axis and so must be the graph of \( f(x) = x^4 \).

They are then prompted to follow the same procedures as in the last experiment:

\[
\text{> simplify(dq(f, x, h));}
\]

\[
4x^3 + 6x^2h + 4xh^2 + h^3
\]

\[
\text{> g := h -> 4*x^3 + 6*x^2*h + 4*x*h^2 + h^3;}
\]

\[
g := h \rightarrow 4x^3 + 6x^2h + 4xh^2 + h^3
\]

\[
\text{> for i to 5 do g(.1^i) od;}
\]
As in the last exercise, the students now have enough experimental evidence to conclude that the derivative of \( f(x) = x^4 \) is \( f'(x) = 4x^3 \).

We’ll now take a break from the computer work and discuss the derivatives of \( x^5 \), \( x^6 \), \( x^7 \), …; we eventually arrive at the conclusion that the derivative of \( f(x) = x^n \) is \( f'(x) = nx^{n-1} \). At this point, the class does some practice problems taking the derivatives of polynomials (they have already learned the derivative of a sum rule and the derivative of a constant times a function rule).

After the students feel comfortable taking the derivatives of polynomials, I’ll suggest that we test whether the rule we just discovered might also be true for negative integers as well. So it’s back to the computers and we start with \( f(x) = x^{-1} \).

\[
\begin{align*}
4x^3 + 0.6x^2 + 0.04x + 0.001 \\
4x^3 + 0.06x^2 + 0.0004x + 1.10^{-5} \\
4x^3 + 0.006x^2 + 4.10^{-5}x + 1.10^{-8} \\
4x^3 + 0.0006x^2 + 4.10^{-7}x + 1.10^{-11} \\
4x^3 + 0.00006x^2 + 4.10^{-9}x + 1.10^{-14}
\end{align*}
\]

We’ll now take a break from the computer work and discuss the derivatives of \( x^5 \), \( x^6 \), \( x^7 \), …; we eventually arrive at the conclusion that the derivative of \( f(x) = x^n \) is \( f'(x) = nx^{n-1} \). At this point, the class does some practice problems taking the derivatives of polynomials (they have already learned the derivative of a sum rule and the derivative of a constant times a function rule).

After the students feel comfortable taking the derivatives of polynomials, I’ll suggest that we test whether the rule we just discovered might also be true for negative integers as well. So it’s back to the computers and we start with \( f(x) = x^{-1} \).

\[
\begin{align*}
f := x \rightarrow x^{-1};
\end{align*}
\]

\[
\begin{align*}
> f := x -> x^(-1); \\
\text{f} := x \rightarrow \frac{1}{x}
\end{align*}
\]

\[
\begin{align*}
> \text{plot( \{f(x), dq(f, x, .1)\}, x = -3 .. 3, -10 .. 10, \text{discont = true});}
\end{align*}
\]
With a bit more difficulty than they had with the polynomial examples, the students are able to determine which of these plots is the graph of \( f(x) = x^{-1} \). They will also know that the other plot, the graph of \( q(f, x, .1) \), is always negative because its graph lies entirely below the \( x \)-axis; however, they will not be so sure as to its algebraic form.

They are now encouraged to follow the methods used in the previous experiments to help determine the derivative of \( f \).

\[ > \text{simplify}(dq(f, x, h)); \]

\[ \frac{-1}{x(x+h)} \]

\[ > g := h \rightarrow -1/(x^2 + x*h); \]

\[ g := h \rightarrow -\frac{1}{x^2 + xh} \]

\[ > \text{for} \ i \ \text{to} \ 5 \ \text{do} \ g(.1^i) \ \text{od}; \]
After evaluating $g(h)$ for several small values of $h$, the students easily ascertain that the derivative of $f(x) = 1/x$ is $f'(x) = -1/x^2$. When this is written with a negative exponent, they also see, at least in this case, that the derivative rule developed for $x^n$ also holds when $n$ is a negative integer. I’ll then ask them to verify this for $f(x) = x^{-2}$.

\[
\begin{align*}
\frac{1}{x^2+1} & - \frac{1}{x^2+0.1} \\
\frac{1}{x^2+0.01} & - \frac{1}{x^2+0.001} \\
\frac{1}{x^2+0.0001} & - \frac{1}{x^2+0.00001}
\end{align*}
\]

\[
\text{After evaluating } g(h) \text{ for several small values of } h, \text{ the students easily ascertain that the derivative of } f(x) = 1/x \text{ is } f'(x) = -1/x^2. \text{ When this is written with a negative exponent, they also see, at least in this case, that the derivative rule developed for } x^n \text{ also holds when } n \text{ is a negative integer. I’ll then ask them to verify this for } f(x) = x^{-2}.
\]

\[\text{> f := x -> x}^{-2};\]

\[f := x \rightarrow \frac{1}{x^2}\]

\[\text{> factor(expand(dq(f, x, h)))};\]

\[\text{> g := h -> -(2*x + h)/((x + h)^2*x^2);}\]

\[g := h \rightarrow -\frac{2x+h}{(x+h)^2 x^2}\]

\[\text{> for i to 5 do g(.1^i) od;}\]
After a little experimenting, the students determine that the derivative of \( f(x) = x^{-2} \) is \( f'(x) = \frac{-2x}{x^4} = -2x^{-3} \) so that the rule seems to hold in this case also. They are now ready to believe that the power rule holds for negative integers as well as positive ones.

To be sure, these procedures do not work in all cases; for example, they don’t work for power functions when the exponent is a rational number nor do they work on logarithm functions. With a little care they can work for exponential functions. In fact, it is an interesting exercise to try these experiments with \( f(x) = e^x \) since the graphs of this function and its difference quotient for \( h = 0.1 \) are nearly coincident.

\[
\begin{align*}
- \frac{2x+.1}{(x+.1)^2 x^2} \\
- \frac{2x+.01}{(x+.01)^2 x^2} \\
- \frac{2x+.001}{(x+.001)^2 x^2} \\
- \frac{2x+.0001}{(x+.0001)^2 x^2} \\
- \frac{2x+.00001}{(x+.00001)^2 x^2}
\end{align*}
\]

\( f := \exp; \)

\( f := \exp \)

\( >\text{plot}( \{ f(x), dq(f, x, .1) \}, x = -3 ..4); \)
I suggest to the class that perhaps the plots might become more coincident for a smaller value of \( h \) and suggest that they replot with \( h = 0.01 \):

```maple
>plot( {f(x), dq(f, x, .01)}, x = -3 ..4);
```

After observing this graph, I ask them to speculate on what this means. A discussion then follows and eventually one or more students suggest that derivative of
$f(x) = e^x$ might be equal to itself. I’ll then ask them to verify this experimentally using the same methods as above.

```maple
> factor(expand(dq(f, x, h)));

\[
\frac{e^x(e^h - 1)}{h}
\]

> g := h -> exp(x)*(exp(h) - 1)/h;

\[
g := h \rightarrow \frac{e^x(e^h - 1)}{h}
\]

for i to 5 do g(.1^i) od;

1.051709180 e^x
1.005016700 e^x
1.000500000 e^x
1.000050000 e^x
1.000000000 e^x

This method also works quite nicely with some of the trigonometric functions.

> f := sin;

\[
f := \sin
\]

> plot( {f(x), dq(f, x, .1)}, x = -2*Pi .. 2*Pi);
Here, they can readily determine the graph of the sine function (they have seen these two graphs together many times previously) so that the other one, which looks remarkably like the cosine function, is the graph of the difference quotient for \( h = 0.1 \). I then have them verify their conclusion experimentally.

\[
\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}
\]

\[
g := h \rightarrow \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}
\]

\[
>\text{for } i \text{ to 5 do } g(.1^i) \text{ od;}
\]
\[-.049958347 \sin(x) + .9983341665 \cos(x)\]
\[-.00499996 \sin(x) + .9999833334 \cos(x)\]
\[-.0005000 \sin(x) + .9999998333 \cos(x)\]
\[-.000050 \sin(x) + .999999983 \cos(x)\]
\[1.00000000 \cos(x)\]