

Problem:

Evaluate the integral $\int_{-1}^3 x^2 + 3x \, dx$ using Riemann Sums.

Solution:

We first subdivide the interval $[-1,3]$ into n subintervals each of length $\Delta x = \frac{b-a}{n} = \frac{3-(-1)}{n} = \frac{4}{n}$

1) If we let x_i be the right-hand endpoint of the i^{th} subinterval, then the corresponding Riemann Sum is

$$\begin{aligned}\sum_{i=1}^n f(x_i) &= \sum_{i=1}^n f(a + i \cdot \Delta x) \Delta x \\ &= \sum_{i=1}^n f\left(-1 + i \cdot \frac{4}{n}\right) \frac{4}{n} \\ &= \frac{4}{n} \sum_{i=1}^n f\left(-1 + i \cdot \frac{4}{n}\right) \\ &= \frac{4}{n} \sum_{i=1}^n \left(-1 + i \cdot \frac{4}{n}\right)^2 + 3 \cdot \left(-1 + i \cdot \frac{4}{n}\right) \\ &= \frac{4}{n} \sum_{i=1}^n \left(-1 + i \cdot \frac{4}{n}\right)^2 + \frac{4}{n} \sum_{i=1}^n 3 \cdot \left(-1 + i \cdot \frac{4}{n}\right) \\ &= \frac{4}{n} \sum_{i=1}^n \left((-1)^2 + 2(-1)i \cdot \frac{4}{n} + i^2 \left(\frac{4}{n}\right)^2 \right) + \frac{4 \cdot 3}{n} \sum_{i=1}^n \left(-1 + i \cdot \frac{4}{n}\right) \\ &= \frac{4}{n} \left(\sum_{i=1}^n (-1)^2 + \sum_{i=1}^n 2(-1)i \cdot \frac{4}{n} + \sum_{i=1}^n i^2 \left(\frac{4}{n}\right)^2 \right) + \frac{4 \cdot 3}{n} \left(\sum_{i=1}^n (-1) + \sum_{i=1}^n i \cdot \frac{4}{n} \right) \\ &= \frac{4}{n} \left(\sum_{i=1}^n 1 + \frac{-2 \cdot 4}{n} \sum_{i=1}^n i + \left(\frac{4}{n}\right)^2 \sum_{i=1}^n i^2 \right) + \frac{4 \cdot 3}{n} \left(-\sum_{i=1}^n 1 + \frac{4}{n} \sum_{i=1}^n i \right) \\ &= \frac{4}{n} \left(n + \frac{-2 \cdot 4}{n} \left(\frac{n(n+1)}{2} \right) + \left(\frac{4}{n}\right)^2 \left(\frac{n(n+1)(2n+1)}{6} \right) \right) + \frac{4 \cdot 3}{n} \left(-n + \frac{4}{n} \left(\frac{n(n+1)}{2} \right) \right) \\ &= 4 - 16 \left(\frac{n+1}{n} \right) + \frac{64}{6} \left(\frac{(n+1)(2n+1)}{n \cdot n} \right) - 12 + 24 \left(\frac{n+1}{n} \right) \\ &= \frac{8(8n^2 + 15n + 4)}{3n^2}\end{aligned}$$

Hence,

$$\int_{-1}^3 x^2 + 3x \, dx = \lim_{n \rightarrow \infty} \frac{8(8n^2 + 15n + 4)}{3n^2} = \frac{64}{3}$$

2) If we let x_i be the left-hand endpoint of the i^{th} subinterval, then the corresponding Riemann Sum is

$$\begin{aligned} \sum_{i=1}^n f(x_i) &= \sum_{i=1}^n f\left(-1 + (i-1)\frac{4}{n}\right) \frac{4}{n} \\ &= \sum_{i=1}^n f\left(-1 + (i-1)\frac{4}{n}\right) \frac{4}{n} \\ &= \frac{4}{n} \sum_{i=1}^n f\left(-1 + (i-1)\frac{4}{n}\right) \\ &= \frac{4}{n} \sum_{i=1}^n \left(-1 + (i-1)\frac{4}{n}\right)^2 + 3 \cdot \left(-1 + (i-1)\frac{4}{n}\right) \\ &= \frac{4}{n} \sum_{i=1}^n \left(-1 + (i-1)\frac{4}{n}\right)^2 + \frac{4}{n} \sum_{i=1}^n 3 \cdot \left(-1 + (i-1)\frac{4}{n}\right) \\ &= \frac{4}{n} \sum_{i=1}^n \left((-1)^2 + 2(-1)(i-1)\frac{4}{n} + (i-1)^2 \left(\frac{4}{n}\right)^2 \right) + \frac{4 \cdot 3}{n} \sum_{i=1}^n \left(-1 + (i-1)\frac{4}{n}\right) \\ &= \frac{4}{n} \left(\sum_{i=1}^n (-1)^2 + \sum_{i=1}^n 2(-1)(i-1)\frac{4}{n} + \sum_{i=1}^n (i-1)^2 \left(\frac{4}{n}\right)^2 \right) \\ &\quad + \frac{4 \cdot 3}{n} \left(\sum_{i=1}^n (-1) + \sum_{i=1}^n (i-1)\frac{4}{n} \right) \\ &= \frac{4}{n} \left(\sum_{i=1}^n 1 + \frac{-2 \cdot 4}{n} \sum_{i=1}^n (i-1) + \left(\frac{4}{n}\right)^2 \sum_{i=1}^n (i-1)^2 \right) + \frac{4 \cdot 3}{n} \left(-\sum_{i=1}^n 1 + \frac{4}{n} \sum_{i=1}^n (i-1) \right) \\ &= \frac{4}{n} \left(n + \frac{-2 \cdot 4}{n} \left(\frac{n(n+1)}{2} - n \right) + \left(\frac{4}{n}\right)^2 \left(\frac{n(n+1)(2n+1)}{6} - 2 \frac{n(n+1)}{2} + n \right) \right) \\ &\quad + \frac{4 \cdot 3}{n} \left(-n + \frac{4}{n} \left(\frac{n(n+1)}{2} - n \right) \right) \\ &= 4 - 16 \left(\frac{n+1}{n} \right) + 32 \left(\frac{1}{n} \right) + \frac{64}{6} \left(\frac{(n+1)(2n+1)}{n \cdot n} \right) - 64 \left(\frac{n(n+1)}{n^3} \right) + \frac{64}{n^2} \\ &\quad - 12 + 24 \left(\frac{n+1}{n} \right) - \frac{48}{n} \\ &= \frac{8(8n^2 - 15n + 4)}{3n^2} \end{aligned}$$

Hence,

$$\int_{-1}^3 x^2 + 3x \, dx = \lim_{n \rightarrow \infty} \frac{8(8n^2 - 15n + 4)}{3n^2} = \frac{64}{3}$$