COGNITIVE TRAITS OF INSTRUCTIONAL TECHNOLOGY LINEAR ALGEBRA

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Advent of new technologies allowed new instructional approaches to mathematical ideas. It is not a surprise that now many areas of mathematics use technology extensively. There is even a new sub-branch of mathematics, computational science that deals solely with the applications of mathematics into computer algorithms. Question then would be what (if any) effect computer assisted instructions have on one's cognition of mathematical concepts, especially abstract ideas commonly known to be difficult to externalize. Despite the common belief that mathematicians do not use visual tools in their work, many researchers including Stylianou(2002) reported mathematicians using visual representations in the early stages of their work mainly to support mental processes, and build intuition. In fact, many mathematicians are already using computer programs to run interactive simulations for the purposes of observing multiple cases at a shorter time frame. Thus, mathematics students too need to be able to carry out multiple tasks at a shorter time frame. This can only become possible with the use of external supports so that learners can effort to direct their attention fully to the abstract aspects of mathematical objects.

Advance technologies allowed us to develop an online environment (see figures 1 and 2 below for snap shots of a module integrated into the environment) that revealed the geometric features of abstract ideas from linear algebra via covering many examples in a short time, in turn allowing our students to observe abstract ideas in multiple examples, thus freeing memory for the processing of observed abstract features. In this paper, we discuss one college student's mental processes as displayed in a one-on-one interview. Interview was conducted right after he completed an activity consisting of a set of mathematics tasks and an interactive online module seen in figures 1 and 2. Mathematics tasks both in the e-activity and the interview were addressing various aspects of the vector space concept, linear independence. See the textbook by Penney (2004) for more information on these topics. These concepts are reported to be very abstract for learners, and they appear to be the main source of many of the learning difficulties in linear algebra (Dogan-Dunlap 2010; 2004; Dogan, 2012; 2001; Dogan, Carrizales and Beaven, 2011; Dorier, Robert, Robinet, and Rogalski, 2000; Dorier and Sierpinska, 2001).

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We will provide a description of the online module while discussing the student's work, also revealing the effect on student cognition. In this paper we will call the student with a pseudo name, SA21. In addition to the work of SA21, we will discuss the qualitative/quantitative data gathered from the interviews of 12 students including SA21. For this portion of the discussion, we will not be referring to the individuals but providing the overall findings revealing similarities and differences in cognition. Seven of the 12 are volunteers from group A; 2 from group B; and 3 from group C. Letters A, B and C refer to the three sections of a first year linear algebra course with emphases on the applications of matrix theory to systems of linear equations, vector spaces and linear transformations. These three groups differed only on the level of integration of an interactive online modules/graphical representations. Group A was fully exposed to the module and the accompanying activities both in and outside class by means of classroom demonstrations and take-home assignments. Group B was exposed to the module activities only as take-home assignments. Group C however was denied of any interaction with the online module activities. Both B and C groups were additionally introduced to static geometric representations during lectures.

RESULTS AND DISCUSSION

Module activity appeared to have influenced SA21 to form a geometric understanding of linear combination, linear independence and plane ideas containing a notion of "dots" in space. Dots are utilized in a module (see figure 2 below). These dots stand for vectors within the applet. Before interviews, SA21's section was given a set of tasks to investigate linear independence of sets of vectors using the tools of the module. In this activity, SA21 entered vectors into the module, and module provided the graphs of these vectors as directed line segments, and the resulting vectors of any linear operation (adding and scalar multiplying) on these vectors as "dots " (see figures 1 and 2). Module is capable of providing multiple dots as the resultant vectors of multiple linear operations. At present, this module can be found at http://www.math.utep.edu:8080/jsp/ under the link "vector Space."

Specifically, SA21 was asked to set the module to get a single resultant vector as a line segment, and observe any linear relations between the initial vectors entered into the module, and the resultant vector to determine linear independence among them. In figure 1, you see three line segments (as vectors) originating from zero vector. Two outer vectors are entered into the module, and the one in the middle provided by the module. The thinner line segment coming out of the vector on the left is there to demonstrate a linear combination, the middle line segment being the direct sum of the two outer vectors. He next was to set the module to observe multiple numbers of resultant vectors from linear combinations of the initial two vectors. See figure 2 for a demonstration of the representation of multiple linear combinations. In this figure, you see three vectors; two are forming a "V" shape, and the third one above the "V" shape. The two forming the "V" shape are used to form the dots as their linear combinations. In fact, if the figure is rotated, one can clearly observe the third vector being located above all the dots. We should note that the module allows 360-rotation in all directions. SA21 was furthermore asked to repeat the two processes multiple times for his own sets of vectors.

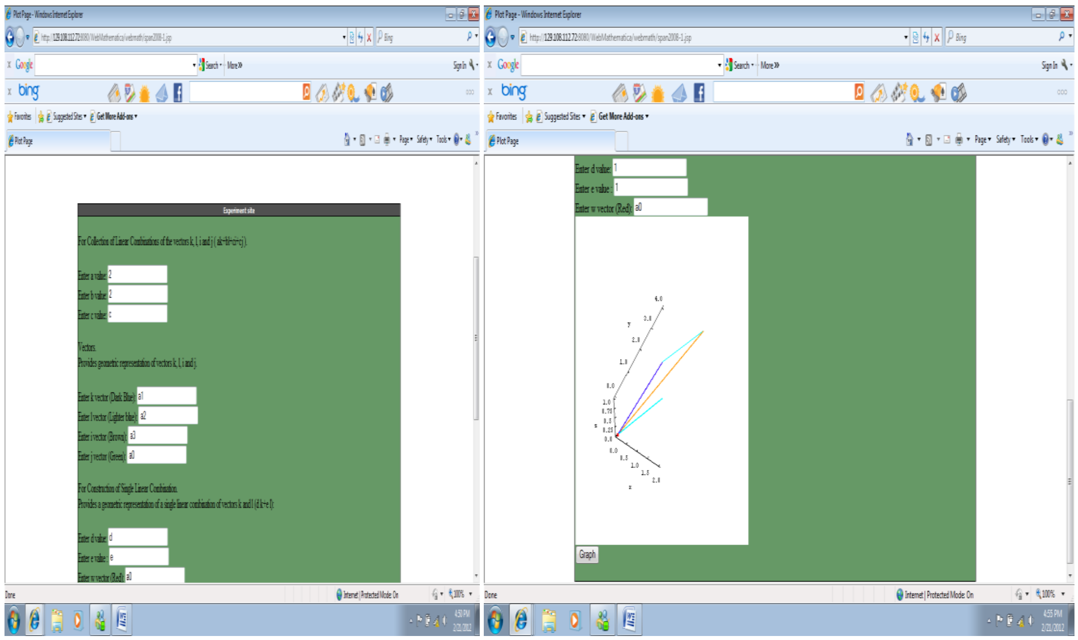

Figure 1. Module view. Single linear combination representation. The middle line segment on the right represent the linear combination of the two outer line segments starting at the origin.

Mental Formations revealed in SA21's Interview

During the interview, when SA21 is asked to respond to the question of whether a set of four vectors (three of which on a plane and one off the plane) is linearly independent, he begins recalling the tasks that he went through recently using the module in figures 1, 2. His response below indicates that he is attempting to reason with his mental constructs he may have formed during the module activity. He is recalling what he would obtain if he were to graph the linear combinations of the vectors given in the question (see figure 2). It is clear in his response below that his mental processes contain the notion of "dots" as linear combinations.

SA21: I have u1, u2, u3, and u4 [four vectors stated in the question where u4 is stated to be off a plane, and the first three on the plane] *we said that these* [u1, u2 and u3] *are on the same plane.*

SA21: So… this one [pointing to u4] is not… this one, should I just write that in here… let's see. I'm thinking from the module [see figures 1 and 2] *it's gonna look like, if we* *were to have it, if we had like this dots it's gonna be a couple of them… and suppose this is the origin 'cause vectors go through the origin. I wanna say we have something like this…*

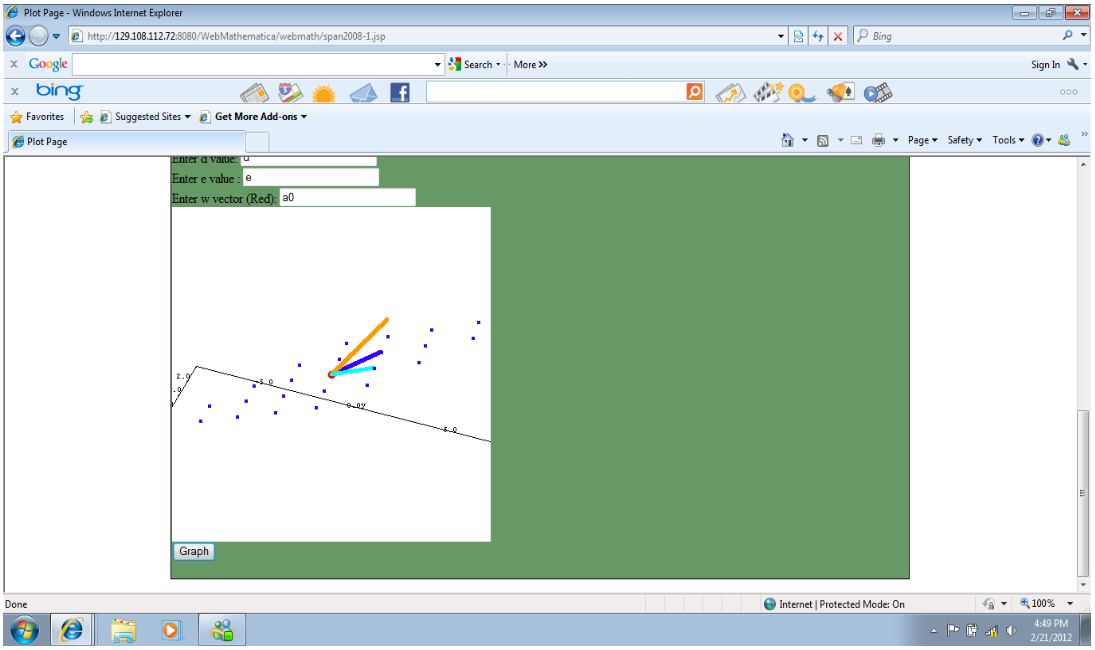

Figure 2. Module view. Dots are the geometric representations of linear combinations of the two line segments (vectors). The middle line segment is a vector that is over the plane formation.

He continues with this mental constructs to address whether u4 would be obtained via "dots" as linear combinations of the other three vectors, u1, u2 and u3 (note that the numerical values of these vectors were not provided). The excerpt below for instance is one where he thinks about linear operations (scalar multiplication and addition of vectors) in the context of the collection of dots, and further wonders whether some of these dots would result in the fourth vector (u4). His use of phrases, "go up" and "jump up," reveals the mental structures he is employing.

SA21: I wanna say independent.

I: Ok, because?

SA21: Because I can't, I can't, I don't think I can form, uh… I can do some like state a scalar, like multiply this one times a scalar add it to this one to get it to ***go up*** *like this…just 'cause these are on the same plane I want to think that they are gonna stay… when we look at the collection they're all just… since they are on a plane, they are just gonna form like this, or somehow, other spots on these dots… but I don't think they'll*

***jump up** to the next… like another dimension…* [student moves pencils/vectors on the table/plane to indicate line segments as vectors and their linear combinations as dots]

The following excerpt furthermore is a testimony to his mental constructs embodying, “dots” and “collection of dots,” as linear combinations and their set. He uses these mental acts to conjecture, “*I just don’t think that somehow we would be able to, to for that to go up*” meaning the dots of linear combinations.

SA21: And, and see if we just continue, so somehow we’re just forming a bunch of other ones and I just don’t think that somehow we would be able to, to… for that to go up…[student moves the pencils/vectors to different positions stating that those are new dots on the plane and they should stay on the same plane].

In the response below, he reiterates his conjecture indicating that none of the linear combinations of two of the three (u1, u2, u3) vectors would get him to the fourth one (u4). His phrase “to get that one,” further reveals his use of graphical entities as linear combinations.

SA21: Oh, and, so, I’m guessing that the… that you just cannot express these two together to get that one.

He further reveals on his responses a mental construct of the collection of dots forming a plane. He uses this mental structure to determine later in the interview the necessary and sufficient condition on the number of vectors to form a plane. In the excerpt below, he is using the phrase, “entire set,” to mean the collection of dots in the form of a plane shape. He combines these mental structures to reason further that he may need more than two vectors to “determine” the fourth vector.

SA21: I think we have two vectors and the only that it should just determine the entire set, maybe... maybe two is just not enough to, to determine [the fourth vector]…

Considering the excerpts discussed so far, one would agree on the type of mental constructs SA21 appeared to have formed due to the module activity. Restating his cognition of the linear independence ideas, we hypothesize that the geometric structures (dots) of linear combinations and their collections (plane) dictate his cognition. In fact, SA21 functions with the geometric aspects of “dots” and “collection of dots” to address the questions of linear independence throughout the interview. We further argue that SA21 mentally holds visual pictures of dots forming a plane shape (see figure 2). He in addition, we believe, informally discovers an abstract idea, the dimension of a space, associated with the minimum number of vectors needed to span a plane. He in fact appears to discover this solely based on his mental constructs due to his work on the module. In summary, as a result of his work in the activity, SA21 appeared to have formed an understanding of:

• a plane as being the collection of dots leading to the ideas of span, and

• that two distinct vectors being sufficient to obtain (via linear operations) all the vectors of a plane leading to the formation of spanning set and basis concepts.

He furthermore appeared to have formed an understanding that:

• "*if a third vector of the initial three vectors is not one of the dots on the plane then the third can never be obtained from the vectors of the plane*".

He in fact explains his understanding using the "dots" notion of linear combinations of vectors. We should restate here that the linear algebra ideas he discussed in his interview were not introduced formally until after the completion of the module activity. Thus, we may safely claim that SA21 formed a geometric understanding of a plane and linear combination ideas mainly due to the work he did within the module activity. Now, let's turn our attention to overall findings from the 12 volunteers.

Cognitive Categories from Three Groups

Table 1 below displays all the cognitive categories emerged as a result of qualitative analysis of the interview responses of 12 volunteers. There are 30 categories emerged. These categories however are not mutually exclusive. Some shares the same responses. After identifying the categories representing cognitive traits, responses in each interview is assigned to appropriate categories, and the number of responses in each category per participant is counted. Next, we obtained average values, per participant for each category, based on the total number of responses from all 30 categories, and furthermore summed up these averages within each group, and obtained another set of averages by dividing the sum with the number of participants in each group. For example, for SA21, we had 35 responses in the first category, "*solution type*." His total for all 30 categories is 604. To obtain an average value for this category, we computed (35/604)*100= 5.794702. We carried out the same process for all 7 students in group A obtaining 12.37113, 0, 5.794702, 0,0, 29.41176, and 14.28571. Next, summing these values, and dividing by 7 gave us the value 8.837616 for the category "*solution type*" for group A. The same process was applied to both B and C groups. We interpreted these averages in order to account for the length of each interview, and the number of participants in each group, thus providing an equal leveling ground for comparison. Figure 3 below shows the bar chart of the percentages of these averages for all the categories and the groups.

Table 1. Categories emerged from a qualitative analysis.

	Codes	Modes
1	ST	Solution type (ST)
2	Sing	Singular/Invertible (Sing)
3	SLE	System of linear equations
4	VO	Vector/Matrix Operation
5	VLC	Verbal linear combination
6	Rn	Rn comparison
7	CLC	Computed Linear Combination
8	RREF	Row reduced Echelon Form
9	IM	Identity Form/Matrix
10	MS	Matrix Size/row vs column comparison
11	NEU	Number of equations vs unknowns
12	DLI	Definition of Linear Independence
13	ZV	Vectors/zero vector
14	ZR	Zero row
15	L	Line
16	PL	Plane/Space
17	LI	Linear Independence
18	DC	Dependent Column
19	V	Variables
20	O	Origin
21	Spn	Span
22	IP	Intersection Point
23	MOM	Multiplicity of Matrices
24	ZEZ	Zero equals zero
25	Difs	Different slopes
26	Norm	Norm
27	COO	Coordinate System
28	Basis	Basis
29	DV	Different vectors
30	TRA	Transpose

Categories, 6,15,16,20,22,25,27 are reflecting cognitive modes with graphical nature. On the categories 6, 15 and 16, group A displays the higher frequency followed by group C. The response rate on the categories 20, 22, 25, and 27 are so low that one may consider them negligible. As for the cognitive modes reflecting non-graphical structures, group B differs significantly from the others on the categories 1, 14, 19 and 28. Table 2 below summarizes the categories in which groups notably deviate. One can deduce from this table that group A contains some of the cognitive categories of geometric nature. Category B however does not contain any of the geometric modes as its dominant mode.

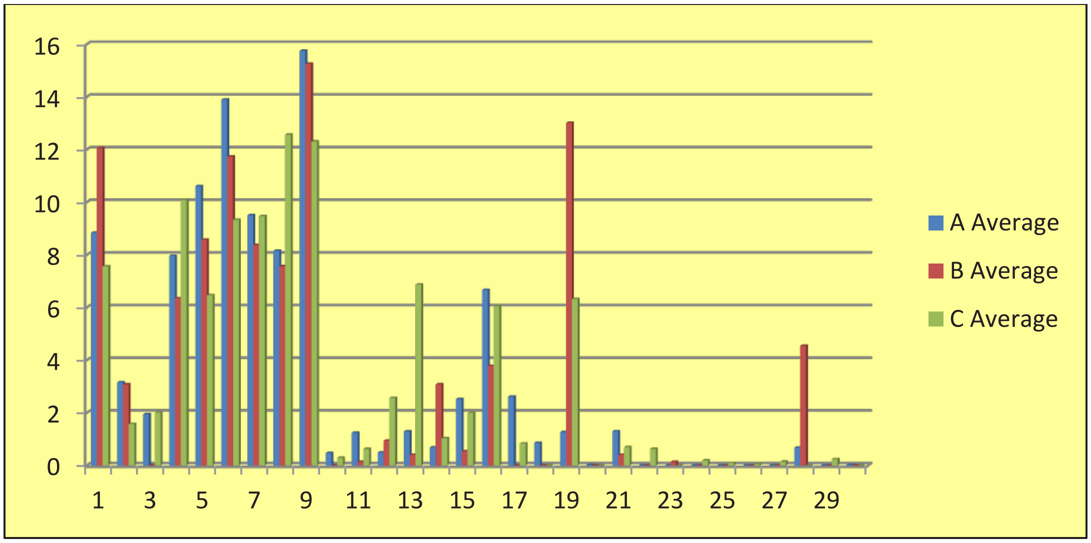

Figure 3. Frequency of responses for each category and groups.

All three groups on the other hand display cognitive categories of both arithmetic and algebraic forms. In addition, one can observe in figure 3 that on two (15 and 16) of the common categories listed on table 2 for groups A and C, group C is trailing group A by a small margin.

Frequency values in table 2 below may be an implication of the influence of our interactive module activity on group A's cognition of linear independence considering that group A displays three cognitive categories (6, 15 and 16) as opposed to none in B and C categories. The response categories furthermore reveal the natures of the geometric modes. Considering that category 6 reflects the comparison of dimensions of Euclidean spaces, and 16 reflects the use of characteristics of planes within the context of spaces, we can say that a notably higher number of responses in A appears to have added in their cognition of linear independence the geometric aspects of Euclidean spaces along with geometric form of vectors as they are located within spaces. To reiterate, noticing the absence of any dominant geometric mode among groups C and B's responses indicates that fully integrated (in and outside class) module activities may be responsible for some of the geometric modes displayed in the student responses of group A.

CONCLUSION

Mathematics is an abstract area for many learners, and ideas in mathematics are difficult to reveal explicitly. Thus, one needs multiple external representations of theoretical concepts to be able to observe the multiple facets that come together to form them. If SA21 were to be asked to sketch the geometric representations of the linear combinations of many sets of vectors, he might have been overwhelmed by the number of sets and sizes of vectors as well as the complexity of computations, thus missing an opportunity to observe many characteristics of vectors, their linear combinations and planes in space. We believe with the ability to sketch multiple sets of vectors, and their linear combinations in an interactive environment, SA21 was able to observe many different examples, and simultaneously do a comparative observation among them leading to him conjecturing an abstract mathematical idea of dimension, "*one needs only two different vectors to form a plane*." SA21 in the response below reiterates our view of the effect of technology on his mental processes. In this excerpt, he states that he was able to observe multiple linear combinations of two vectors given in our module activity. Furthermore, he says, "*I could not get this to get this one* [meaning a third vector of the plane again given in his module task]." Clearly, SA21 was able to crunch out the geometric representations of many vectors, and their linear combinations multiple times in a short time frame. Since our module did the computational and graphical tasks for him, it gave him time to focus solely on the mathematical relations between the geometric representations of vectors and their linear combinations leading to his observation of "*vectors off a plane are not the result of any linear combinations…*"

SA21: I'm thinking in one of the modules there was one of those questions, I'm trying to think which one was that you gave us, I can't remember the numbers but it had… when I would rotate it then it would do it, like that on that same plane and this one was just off [see figure 2 above] and I tried so many numbers [as coefficient values for linear operations among the given vectors], and I could not get this to get this one… [Student takes two pencils/vectors to state that he couldn't multiply the first by a scalar to get the second one].

Table 2. Numerical Values of the Categories with higher frequency with frequency values above one point.

Groups	A	B	C
	2	1	3
	3	2	4
	5	14	7
	6	19	8
	7	28	12
	9		13
	11		
	15		
	16		
	17		
	21		

Our findings of 12 volunteers from three different sections of the same course with varying level of technology integration further support the cognitive modes SA21 displayed on his responses. We found notable differences on the type of categories each group displayed. In fact, group, A, with higher level of module integration displayed more categories with geometric nature. We also found that the same group did not lack on the categories of algebraic and arithmetic modes, implying that the module integration in group A may not only result in the formation of geometric categories, but it also did not stop students from forming other categories of algebraic and arithmetic modes.

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