EXPLORING SEVERAL OPEN PROBLEMS IN ELEMENTARY NUMBER THEORY WITH CAS TECHNOLOGY

Jay L. Schiffman
Rowan University
201 Mullica Hill Road, Glassboro, NJ 08028-1701
e-mail: schiffman@ rowan.edu

ABSTRACT: Elementary Number Theory is a fascinating branch of mathematics that furnishes a treasure-trove of easily posed conjectures that have escaped resolution throughout the rich history of mathematical lore. Some typical open problems in the field include prime decades and even perfect numbers. In addition to discussing these problems in detail, we will furnish programs amenable to the TI-89/VOYAGE 200 calculator to enable participants to obtain a better foothold of the richness of the mathematics encompassed in these problems and hopefully discover new mathematical insights with the aid of our technological tools.

We initiate our discussion with the definition of a prime decade. A ***prime decade*** is a set of ten integers from n_0 to n_9 (where n is a positive integer of any length with the subscripts being the last digits) such that n_1, n_3, n_7, *and* n_9 are all primes. (Of course, all primes with the exception of 2 and 5 terminate in one of the digits 1, 3, 7, and 9.) Thus in the sequence of ten integers one has a pair of twin primes twice. For example, 11, 13, 17, and 19 constitutes a prime decade while the next one is 101, 103, 107, and 109. The question that remains unresolved is whether there are infinitely many prime decades. We next generate two user-defined programs incorporating the idea of a prime decade below:

Program For Prime Decades:

Define primdeca (n) = when (is Prime (n) = true and is Prime $(n+2)$ = true and is Prime $(n+6)$ = true and is Prime $(n+8)$ = true and $\mathrm{mod}(n,10)=1$, true, false)

Program To List The First, Second, Third, and Fourth Elements In A Prime Decade:
If we desire to generate the first, second, third, and fourth outputs in the prime decade, proceed as follows:

Define primdec0 (n) = when (is Prime (n) = true and is Prime $(n+2)$ = true and is Prime $(n+6)$ = true and is Prime $(n+8)$ = true and $\mathrm{mod}(n,10)=1$, n, false)

Define primdec2 (n) = when (is Prime (n) = true and is Prime $(n+2)$ = true and is Prime $(n+6)$ = true and is Prime $(n+8)$ = true and $\mathrm{mod}(n,10)=1$, $n+2$, false)

Define primdec6 (n) = when (is Prime (n) = true and is Prime $(n+2)$ = true and is Prime $(n+6)$ = true and is Prime $(n+8)$ = true and $\mathrm{mod}(n,10)=1$, $n+6$, false)

Define primdec8 (n) = when (is Prime (n) = true and is Prime $(n+2)$ = true and is Prime $(n+6)$ = true and is Prime $(n+8)$ = true and $\mathrm{mod}(n,10)=1$, $n+8$, false)

We next determine if a prime decade is formed starting with the primes 281, 1481, and 1006331. We use the program primdeca. See **FIGURE 1** below:

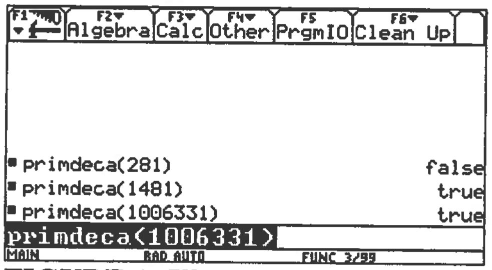


FIGURE 1: Illustration of the primdeca program.

Conclude that the prime 281 does not initiate a prime decade; for while 281 and 283 are indeed primes, 287 and 289 are not. $\left(287=7\cdot 41 \textit{ and } 289=17^2\right)$. On the other hand, the decade starting with the prime 1481 is indeed a prime decade as is the decade starting with the prime 1006331. Alternatively, one could check these assertions utilizing the factor or is prime options. There are many interesting facts about prime decades. To find the next two prime decades after the prime decade $\{11,13,17,19\}$, we employ the VOYAGE 200 in FUNCTION MODE with the following seven functions in **FIGURE 2,** the TABLE SETUP in **FIGURE 3** and TABLE in **FIGURES 4-5** below:

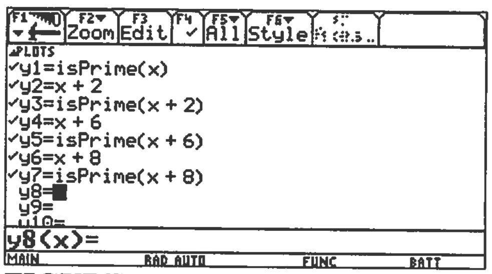


FIGURE 2: The Function Inputs.

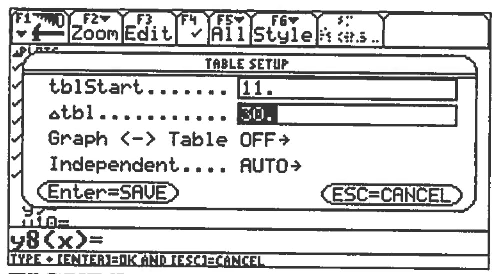
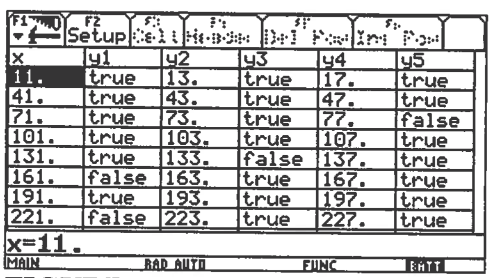
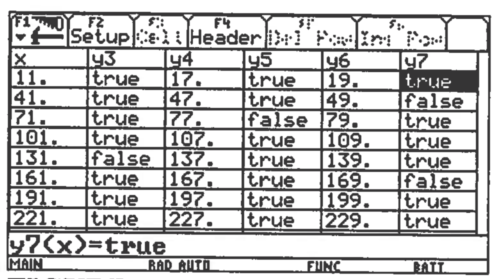


FIGURE 3: The Table Setup.



x	y1	y2	y3	y4	y5
11.	true	13.	true	17.	true
41.	true	43.	true	47.	true
71.	true	73.	true	77.	false
101.	true	103.	true	107.	true
131.	true	133.	false	137.	true
161.	false	163.	true	167.	true
191.	true	193.	true	197.	true
221.	false	223.	true	227.	true

FIGURE 4: The Table Revealed.



x	y3	y4	y5	y6	y7
11.	true	17.	true	19.	true
41.	true	47.	true	49.	false
71.	true	77.	false	79.	true
101.	true	107.	true	109.	true
131.	false	137.	true	139.	true
161.	true	167.	true	169.	false
191.	true	197.	true	199.	true
221.	true	227.	true	229.	true

FIGURE 5: The Table Revealed.

One can analyze the table setup in **FIGURE 3** above as follows where p is a prime:

If $\{p, p+2, p+6, p+8\}$ constitutes a prime decade, then clearly each of the integers in the set below is divisible by 3 and hence is not prime:

$\{p+1, p+4, p+7, p+10, p+13, p+16, p+19, p+22, p+25, p+28\}$

(In any set of three consecutive integers, one of them is always divisible by 3.) Hence if the set $\{p, p+2, p+6, p+8\}$ comprises a prime decade, then each of the following sets cannot constitute a prime decade:

1. $\{p+10, p+12, p+16, p+18\}$
2. $\{p+20, p+22, p+26, p+28\}$

Hence there must be a minimal distance of 30 between prime decades. Using ideas involving the divisibility of an integer by 7 will show that no pair of prime decades can have a distance of 60 between them. The details are left to the interested reader. Examining **FIGURES 4-5** above, we see that the next two prime decades after $\{11,13,17,19\}$ *are*: $\{101,103,107,109\}$ *and* $\{191,193,197,199\}$.

It can be shown via an exhaustive search that the initial prime decades at a distance of 30 from each other are

$\{1006301, 1006303, 1006307, 1006309\}$ *and* $\{1006331, 1006333, 1006337, 1006339\}$.

What is most fascinating is that there are no primes between the primes 1006309 and 1006331. Thus one has these eight primes in the two prime decades being consecutive in nature. See **FIGURE 6** below using the Next Prime program found on P. 435 of the TI-89 guidebook courtesy of Texas Instruments, Dallas, TX. (1998). The Program is as follows:

Program For The Next Prime After A Given Positive Integer:
Define next Prim (n) = Func: Loop: $n+1 \rightarrow n$: If is Prime (n): Return n:
End Loop: End Func

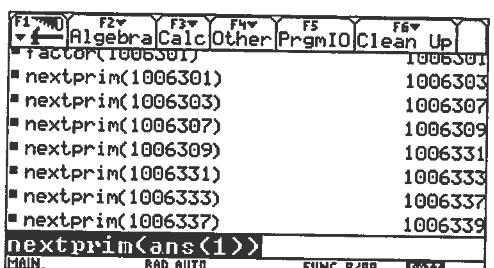


FIGURE 6: Illustration of the Next Prime Program.

We next illustrate our program to secure the first, second, third, and fourth elements in each of the prime decades starting with the respective primes 1871, 3461, and 1006301. See **FIGURES 7-9** below. Our programs are primdec0, primdec2, primdec6 and primdec8 respectively.

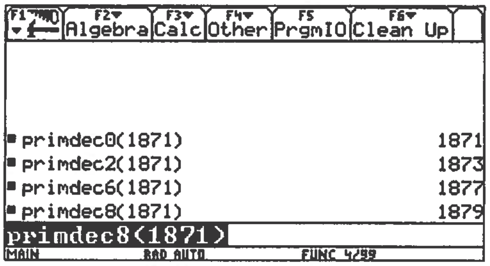


FIGURE 7: primdec programs shown.

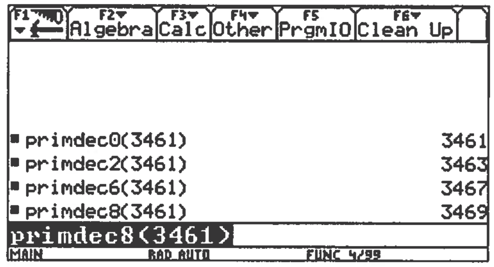


FIGURE 8: primdec programs shown.

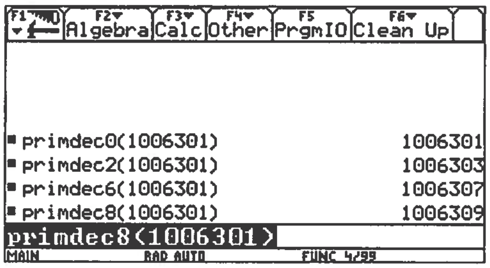


FIGURE 9: primdec programs shown.

Ideas involving even perfect numbers have fascinated mathematicians since Greek antiquity. Even perfect numbers such as 6, 28, and 496 in which the sum of all the aliquot (proper) divisors of the number coincide with the number were known since the time of Euclid. The question as to whether infinitely many such numbers exist remains open and currently only 46 have been secured including two recently via the celebrated GIMPS Project. The following easily generated user-defined programs can be generated:

1. Program For Mersenne Numbers:
Define mersenne (n) =when (is Prime (n) = true, $2^n - 1$, false)
2. Program For Mersenne Primes:
Define mersepri (n) =when (is Prime (n) =true and is Prime ($2^n - 1$) =true, $2^n - 1$, false)
3. Program For Even Perfect Numbers:
Define perfect (n) =when (is Prime (n) =true and is Prime ($2^n - 1$) =true, $(2^n - 1) \cdot (2^{(n-1)})$, false)

We use the program mersenne to determine the associated Mersenne numbers corresponding to the primes 13, 19, and 23 respectively. See **FIGURE 10** below:

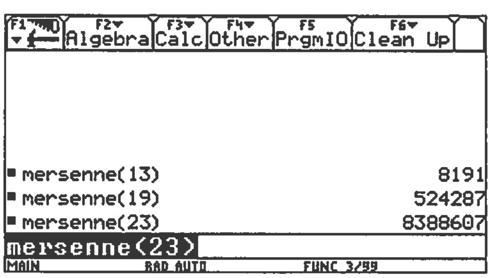


FIGURE 10: Mersenne Program Illustrated.

Any positive integer n of the form $n = 2^p - 1$ where p is prime is known as a Mersenne number.

In order to determine whether a prime p leads to a Mersenne prime, we must check to see whether $n = 2^p - 1$ is likewise prime. We use the program mersepri as in **FIGURE 11** below:

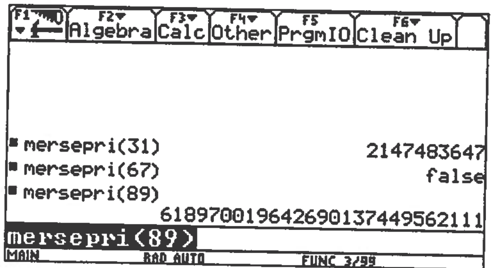


FIGURE 11: Mersenne Prime Program Illustrated.

We conclude that while $2^{31} - 1$ *and* $2^{89} - 1$ are indeed primes, $2^{67} - 1$ is composite. The prime factorization of this integer is given in **FIGURE 12** below:

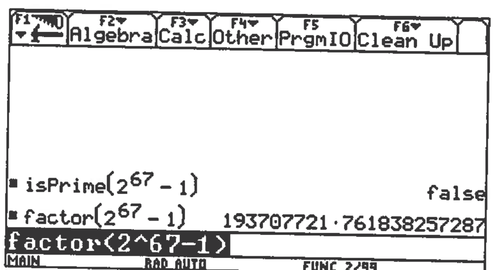


FIGURE 12: Illustrating the is Prime and Factor Commands on the TI-89 / VOYAGE 200 Handhelds.

Euclid proved that if a positive integer n is of the form $n = \left(2^{p-1}\right)\cdot\left(2^p - 1\right)$, where both p and $2^p - 1$ are primes, then n is an even perfect number. Euler proved the converse in the eighteenth century. We check to see if the primes 17, 37, and 61 lead to even perfect numbers using our program perfect. In order to achieve our goal, we use the program perfect to verify that the primes 17 and 61 lead to even perfect numbers while the prime 37 does not. The results are given in **FIGURE 13** below:

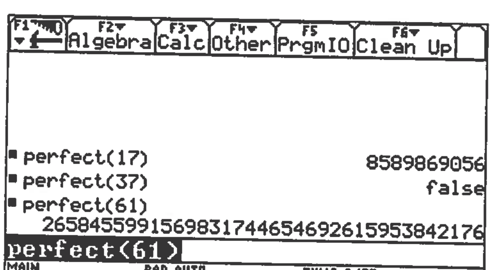


FIGURE 13: Illustrating the Perfect Number Program.