EXPLORING SEVERAL OPEN PROBLEMS IN ELEMENTARY NUMBER THEORY WITH CAS TECHNOLOGY

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ABSTRACT: Elementary Number Theory is a fascinating branch of mathematics that furnishes a treasure-trove of easily posed conjectures that have escaped resolution throughout the rich history of mathematical lore. Some typical open problems in the field include prime decades and even perfect numbers. In addition to discussing these problems in detail, we will furnish programs amenable to the TI-89/VOYAGE 200 calculator to enable participants to obtain a better foothold of the richness of the mathematics encompassed in these problems and hopefully discover new mathematical insights with the aid of our technological tools.

We initiate our discussion with the definition of a prime decade. A prime decade is a set of ten integers from \( n_0 \) to \( n_9 \) (where \( n \) is a positive integer of any length with the subscripts being the last digits) such that \( n_1, n_3, n_7, \) and \( n_9 \) are all primes. (Of course, all primes with the exception of 2 and 5 terminate in one of the digits 1, 3, 7, and 9.) Thus in the sequence of ten integers one has a pair of twin primes twice. For example, 11, 13, 17, and 19 constitutes a prime decade while the next one is 101, 103, 107, and 109. The question that remains unresolved is whether there are infinitely many prime decades. We next generate two user-defined programs incorporating the idea of a prime decade below:

Program For Prime Decades:
Define primdeca \( (n) \) = when (is Prime \( (n) \) = true and is Prime \( (n+2) \) = true and is Prime \( (n+6) \) = true and is Prime \( (n+8) \) = true and mod\( (n,10) \)=1, true, false)

Program To List The First, Second, Third, and Fourth Elements In A Prime Decade:
If we desire to generate the first, second, third, and fourth outputs in the prime decade, proceed as follows:
Define primdec0 \( (n) \) = when (is Prime \( (n) \) = true and is Prime \( (n+2) \) = true and is Prime \( (n+6) \) = true and is Prime \( (n+8) \) = true and mod\( (n,10) \)=1, \( n, \) false)
Define primdec2 \( (n) \) = when (is Prime \( (n) \) = true and is Prime \( (n+2) \) = true and is Prime \( (n+6) \) = true and is Prime \( (n+8) \) = true and mod\( (n,10) \)=1, \( n+2, \) false)
Define primdec6 \( (n) \) = when (is Prime \( (n) \) = true and is Prime \( (n+2) \) = true and is Prime \( (n+6) \) = true and is Prime \( (n+8) \) = true and mod\( (n,10) \)=1, \( n+6, \) false)
Define $\text{primdec8}(n) = \text{when (is Prime } (n) = \text{true and is Prime } (n+2) = \text{true and is Prime } (n+6) = \text{true and is Prime } (n+8) = \text{true and mod}(n,10)=1, \, n+8, \, \text{false})$

We next determine if a prime decade is formed starting with the primes 281, 1481, and 1006331. We use the program primdeca. See FIGURE 1 below:

![Figure 1: Illustration of the primdeca program.](image)

Conclude that the prime 281 does not initiate a prime decade; for while 281 and 283 are indeed primes, 287 and 289 are not. ($287 = 7 \cdot 41 \text{ and } 289 = 17^2$). On the other hand, the decade starting with the prime 1481 is indeed a prime decade as is the decade starting with the prime 1006331. Alternatively, one could check these assertions utilizing the factor or is prime options. There are many interesting facts about prime decades. To find the next two prime decades after the prime decade $\{11,13,17,19\}$, we employ the VOYAGE 200 in FUNCTION MODE with the following seven functions in FIGURE 2, the TABLE SETUP in FIGURE 3 and TABLE in FIGURES 4-5 below:

![Figure 2: The Function Inputs.](image)

![Figure 3: The Table Setup.](image)

![Figure 4: The Table Revealed.](image)

![Figure 5: The Table Revealed.](image)

One can analyze the table setup in FIGURE 3 above as follows where $p$ is a prime:
If \( \{p, p+2, p+6, p+8\} \) constitutes a prime decade, then clearly each of the integers in the set below is divisible by 3 and hence is not prime:
\[ \{p+1, p+4, p+7, p+10, p+13, p+16, p+19, p+22, p+25, p+28\} \]
(In any set of three consecutive integers, one of them is always divisible by 3.) Hence if the set \( \{p, p+2, p+6, p+8\} \) comprises a prime decade, then each of the following sets cannot constitute a prime decade:
1. \( \{p+10, p+12, p+16, p+18\} \)
2. \( \{p+20, p+22, p+26, p+28\} \)

Hence there must be a minimal distance of 30 between prime decades. Using ideas involving the divisibility of an integer by 7 will show that no pair of prime decades can have a distance of 60 between them. The details are left to the interested reader. Examining FIGURES 4-5 above, we see that the next two prime decades after \( \{11, 13, 17, 19\} \) are: \( \{101, 103, 107, 109\} \) and \( \{191, 193, 197, 199\} \).

It can be shown via an exhaustive search that the initial prime decades at a distance of 30 from each other are
\( \{1006301, 1006303, 1006307, 1006309\} \) and \( \{1006331, 1006333, 1006337, 1006339\} \).

What is most fascinating is that there are no primes between the primes 1006309 and 1006331. Thus one has these eight primes in the two prime decades being consecutive in nature. See FIGURE 6 below using the Next Prime program found on P. 435 of the TI-89 guidebook courtesy of Texas Instruments, Dallas, TX. (1998). The Program is as follows:

Program For The Next Prime After A Given Positive Integer:
Define next Prim \( (n) \) = Func: Loop: \( n+1 \to n \): If is Prime \( (n) \): Return \( n \):
End Loop: End Func

FIGURE 6: Illustration of the Next Prime Program.

We next illustrate our program to secure the first, second, third, and fourth elements in each of the prime decades starting with the respective primes 1871, 3461, and 1006301. See FIGURES 7-9 below. Our programs are primdec0, primdec2, primdec6 and primdec8 respectively.
Ideas involving even perfect numbers have fascinated mathematicians since Greek antiquity. Even perfect numbers such as 6, 28, and 496 in which the sum of all the aliquot (proper) divisors of the number coincide with the number were known since the time of Euclid. The question as to whether infinitely many such numbers exist remains open and currently only 46 have been secured including two recently via the celebrated GIMPS Project. The following easily generated user-defined programs can be generated:

1. Program For Mersenne Numbers:
   Define \( mersenne (n) = \text{when (is Prime (n) = true, } 2^n - 1, \text{ false) } \)

2. Program For Mersenne Primes:
   Define \( mersepri (n) = \text{when (is Prime (n) =true and is Prime (} 2^n - 1 \text{) =true, } 2^n - 1, \text{ false) } \)

3. Program For Even Perfect Numbers:
   Define \( perfect (n) = \text{when (is Prime (n) =true and is Prime (} 2^n - 1 \text{) =true, } (2^n - 1) \cdot (2^{(n-1)}) \text{, false) } \)

We use the program mersenne to determine the associated Mersenne numbers corresponding to the primes 13, 19, and 23 respectively. See FIGURE 10 below:
Any positive integer \( n \) of the form \( n = 2^p - 1 \) where \( p \) is prime is known as a Mersenne number.

In order to determine whether a prime \( p \) leads to a Mersenne prime, we must check to see whether \( n = 2^p - 1 \) is likewise prime. We use the program mersepri as in FIGURE 11 below:

![Figure 11: Mersenne Prime Program Illustrated.](image)

We conclude that while \( 2^{31} - 1 \) and \( 2^{89} - 1 \) are indeed primes, \( 2^{67} - 1 \) is composite. The prime factorization of this integer is given in FIGURE 12 below:

![Figure 12: Illustrating the is Prime and Factor Commands on the TI-89 / VOYAGE 200 Handhelds.](image)

Euclid proved that if a positive integer \( n \) is of the form \( n = (2^p - 1)(2^p - 1) \), where both \( p \) and \( 2^p - 1 \) are primes, then \( n \) is an even perfect number. Euler proved the converse in the eighteenth century. We check to see if the primes 17, 37, and 61 lead to even perfect numbers using our program perfect. In order to achieve our goal, we use the program perfect to verify that the primes 17 and 61 lead to even perfect numbers while the prime 37 does not. The results are given in FIGURE 13 below:

![Figure 13: Illustrating the Perfect Number Program.](image)