# ELECTRIC CIRCUITS AND DIFFERENTIAL EQUATIONS: AUDIO DATA COLLECTION

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**Abstract:** Standard computer audio hardware and software provides an inexpensive signal generator and recorder to collect data from real circuits. This presentation is intended to demonstrate that this is can be extremely inexpensive (materials cost less than $10) and straightforward. Data is generated and collected using the free audio package Audacity [1] that runs under Windows, Mac, and Linux. Collected data is graphed and analysed using a Computer Algebra System or Spreadsheet. Mathematica will be used to demonstrate the analysis. Physical circuit elements (resistors, capacitors, and inductors) are compared to the mathematical idealizations commonly introduced in Ordinary Differential Equations (ODE) courses. Standard exponential decay and rise curves presented in Calculus will be measured in simple circuits with Inductors and Capacitors. The standard Inductor, Resistor, and Capacitor RC, LR, and resonant LRC circuits are demonstrated and compared to the theoretical response presented in differential equations courses.

**Introduction:** The standard first example of a physical system modelled by a linear, second-order, constant-coefficient ODE is a mass-spring system driven by an external force. However, it is difficult to make a good physical classroom demonstration with quantitative data collection of a mass-spring system for two reasons. The first reason is that the physics of a demonstration scale mass-spring system is substantially more complex than represented in most ODE texts: the springing motion tends to couple with the two dimensional pendulum modes of the system; air-resistance on this scale is quadratic not linear; most springs are strongly non-linear in compression; and it is difficult to reduce the mainly internal spring damping. The second reason is that it is difficult to accurately record the motion and apply controlled forces.

The standard second example of a physical system modelled by a linear, second-order, constant-coefficient ODE is an LRC (Inductor, Resistor, Capacitor) circuit driven by an external voltage supply. In contrast to the mass-spring system, it is easy to make a good physical classroom demonstration with quantitative data collection of an LRC circuit: Kirchoff's circuit equations accurately describe simple physical circuits; resistors, capacitors, and inductors are well-described by their mathematical idealizations; simple electronic components are extremely inexpensive and available with a very-broad range of resistances and capacitances; and it is easy and inexpensive to supply a proscribed external voltage and record the experiment using any laptop or desktop computer.

The reason for this ordering (mass-spring systems first electrical-circuits second) is that students are more familiar with and better able to visualize the motion of a mass on a spring than simple circuits. However, students in our engineering ODE course are frequently taking or have just completed an introductory circuits class and I believe the ease with which good quantitative data can be collected and the connections that can be demonstrated between the cicuits class and ODE is more important than this familiarity.

The primary goal of this paper is to describe how to use the free audio package Audacity [1] (that runs under Windows, Mac, and Linux) to drive and record simple demonstration electrical circuits. The focus is on the circuit examples commonly presented in ODE texts i.e. the transient and long-term sinusoidal response of Resistor-Capacitor (RC), Inductor-Resistor (LR), and Inductor-Resistor-Capacitor (LRC) circuits. Due to space constraints we restrict data presented to the LR circuit and focus on the potentially unfamiliar mechanics of building ta circuit on breadboard and connecting and recording the data.

Minimal information is provided on circuit construction and sourcing of supplies. Commercial resistors and capacitors with suitable parameter value are inexpensive and easily available from any electrical supply while commercial high parameter-value inductors are expensive, and physically heavy. Fortunately, any electrical motor (functioning or not) is a high-parameter value inductor with weight being a fairly good indicator of inductance. Capacitors, resistors, and breadboard connectors for these experiments were purchased as surplus from Electronics Goldmine [2] with a total cost of less than $10. The inductor used is a substantial alternating current motor from a very old computer printer with three (red, blue, and yellow) power leads and measured inductance of 0.12H, 0.64H, or 0.76H depending which pair of leads is used.

Qualitative and quantitative classroom demonstrations serve different purposes. A qualitative demonstration in a math class is usually intended to help fix the physical situation in the students mind and demonstrate that the model (in our case ODE) predicts behavior broadly-consistent with the demonstration. A quantitative demonstration (with the additional goal of analyzing recorded data and demonstrating detailed agreement with the numerical predictions of the model) is usually intended to convince students that the model is directly usefull. The data collected in the experiments in this paper are analyzed directly using elementary techniques in Mathematica [3].

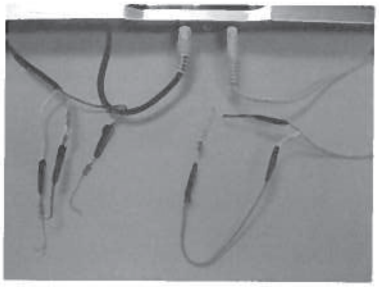
Standard exponential decay and rise data from RC and LR crcuits will be collected and analyzed as will the long-time sinusoidal response (over a broad range of frequencies) in RC, LR, and resonant LRC circuits. I hope that the ODE analysis presented here (which differs significantly from the presentation in a typical circuits course) convinces students that the ODE circuit equations are practical and reinforces the transform techniques common in the circuits presentations.

**Connectors:** Laptops and desktop computers all have stereo headphone-out and mic-in sockets. These circuits are not easy to damage because they need to be able to drive a broad range of headphones and record from a broad range of microphones. Roughly speaking: The headphone driver is anticipating a relatively high resistance and provides a safe voltage of between +1.5V and -1.5V over an audio frequency range while the mic-in is is anticipating milli-volt levels and has a relatively high resistance. The output level of the headphone socket and amplification of the mic-in socket are easily controlled with the computer volume and recording control panel.

Practically this means that:
1) You should always connect a substantial resistor in series with your circuit and the headphone voltage source to avoid distortion from shorting the headphone circuit.
2) You should always connect substantial resistors between the circuit and the left and right mic-in cables to avoid clipping in the microphone circuits.
3) You should restrict attention to audio-frequencies (normally quoted as 20Hz-20MHz) to stay in the microphone design range and avoid triggering any of the switching circuitry in the audio card.
4) It is very-very difficult to damage the mic-in or headphone circuits. On most computers the sockets (which both take the same Tip-Ring-Sleeve (TRS)) are adjacent and no common misconnections will cause any harm. Specifically, neither shorting the headphone circuit or clipping the micrphone circuit by directly connecting the headphone to the mic cause any harm on any of the computers I have tested.
5) The voltages from the headphone socket are completely harmless.

To make the connectors you cut one audio-audio connector in the middle with a pair of scissors and strip back the (in my case black) plastic of the outer cable to expose three conductors a visible copper shield wire and two inner (in my case red and black plastic wrapped) wires. The shield is the computer ground with the red and black being the left and right audio channels.

We are only using one driver signal so strip and twist the left and right channels (red and black wires) together on one half to make a mono connector and attatch a substantial resistor to avoid any risk of shorting the headphone circuit. Plug this into the headphone output socket. We are using two input signals so strip the left and right channels (red and black wires) on the other half and attatch identical substantial resistors (to avoid any risk of clipping in the microphone circuit) to the red and back wires. Plug this into the microphone input socket. Finally, twist the two bare copper shield wires together.



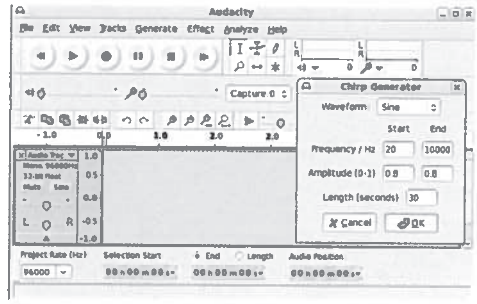



Figure 1: Connectors & Figure 2: Audacity Chirp pop-up menu.

**Signal Generation:** We need two different signals for our exeriment. The first signal is a sinusoidal frequency sweep to demonstrate the steady state behavior of our circuits under different frequencies. Selecting Chirp under the Generate menu produces the pop-up window shown in Figure 2.

**RC and LR Circuits:** An RC and an LR circuit (left and right) are shown on a breadboard (used to keep circuits tidy with automatic connections within each horizontal strip of five sockets and within both complete strips of vertical sockets) in Figure 3. For teh experiments we use a recycled high-quality audio capacitor with a (fairly high) rated capacitance of $4.7\mu$F and the motor inductor described above. A multi-meter gives the values of the inductance and capacitance as 0.76H $4.8\mu$F respectively with a small resistance of the inductor coils of $157\Omega$ .

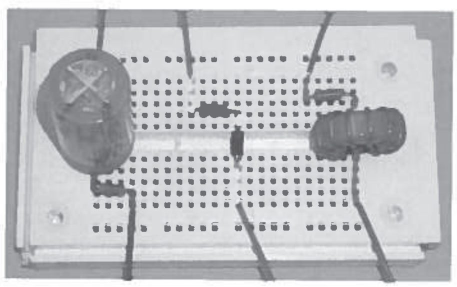


Figure 3: RC (left) and LR (right) Circuits. Highlighting shows the breadboard cruit paths.

**RC and LR Experiments:** Pushing the round record button in Audacity plays the track while recording the voltages from the LR circuit in Figure 3 and produces the stereo track shown at the bottom of Figure 4. The left track is the total voltage (above computer ground) across the capacitor and resistor the right track is the voltage across the resistor alone. As time goes on and the frequency increases the voltage across the resistor and capacitor is constant while the voltage across the resistor decreases. The RC circuit is record in a similar manner by rearranging the mic inputs. The output is essentially the opposit with the voltage across the capacitor decreasing as the frequency increases.

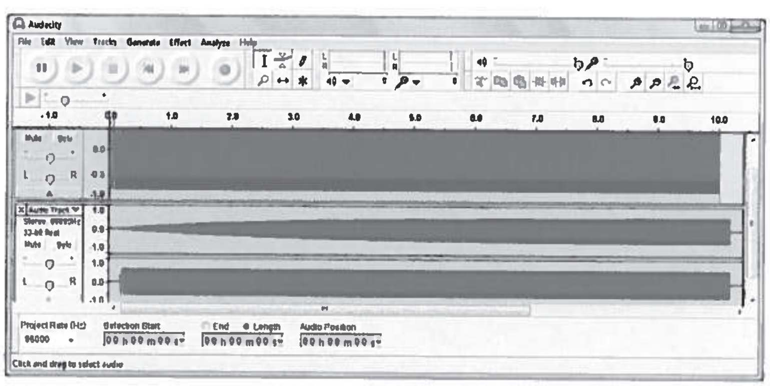


Figure 4: LR Frequency Sweep Experiment.

**LRC Circuits:** Serial and parallel LRC circuits, shown in Figure 5, are easy to buld and produce more interesting results. For the serial circuit, as time goes on and the frequency increases the voltage across the resistor and inductor and capacitor is constant while the voltage across the capacitor increases and then decreases.

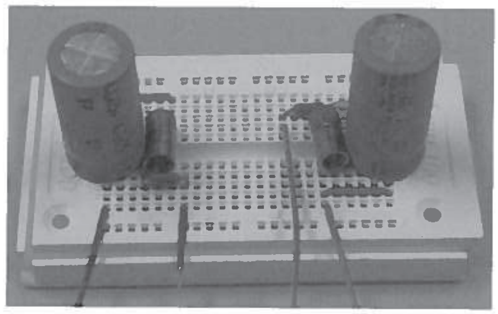


Figure 5: LRC Circuits. Series on the left parallel on right.

The remainder of the paper explains why this happens using the language of ODEs, demonstrates that the LR experimental amplitudes and phase shifts match the ODE model predictions well, and explains that the others do as well.

**Model ODE:** The basic model equation (see for example sections 3.4, and 4.7 of the text by Polking, Bogess and Arnold [5] for a complete description including many algebraic details) describing the charge $q(t)$ on the capacitor in a sinusoidally driven (with amplitude $A$ and frequency $\omega$) series LRC circuit with the inductance L, resistance R, and capacitance C is

(1) $$L q''(t) + R q'(t) + \frac{1}{C} q = A \sin(\omega t).$$

The first, second, and third terms give the voltage drops across the inductor, resistor and capacitor respectively which (by Kirchoff's first law) equals the driving voltage. For the RC circuit there is no inductor and $L = 0$ leaving the simple, constant-coefficient, first-order equation.

(2) $$R q'(t) + \frac{1}{C} q = A \sin(\omega t).$$

with solution (given by standard solution techniques)

(3) $$q(t) = c_{rc} e^{-\lambda_{rc} t} + A G_{cr} \sin(\omega t - \phi_{rc})$$

where the decay constant $\lambda_{rc} = 1/(CR)$, the gain $G_{rc}$ and phase shift $\phi_{rc}$ are given by

(4) $$G_{rc} = \left(\lambda_{rc}^2 + \omega^2\right)^{-1/2} / R \text{ and } \phi_{rc} = \arccos\left(\lambda_{rc}\left(\lambda_{rc}^2 + \omega^2\right)^{-1/2}\right),$$

and the constant $c_{rc}$ is determined by the initial charge on the capacitor. For the LR circuit the capacitor is missing and dropping the capacitance term gives the first order equation for the current through the circuit $i(t) = q'(t)$

(5) $$L i'(t) + R i(t) = A \sin(\omega t)$$

with solution

(6) $\quad i(t) = c_{lr}\, e^{-\lambda_{lr} t/\tau_{rl}} + A\, G_{lr} \sin(\omega\, t - \phi_{lr})$

where the decay constant $\lambda_{lr} = R/L$, and the gain $G_{lr}$ and phase shift $\phi_{lr}$ are given by substituting $\lambda_{lr}$ for $\lambda_{rc}$ in expreesions (4) and the constant $c_{lr}$ is determined by the initial current in the circuit. The solution of the general LRC circuit (in the physically-interesting, underdamped case when $R < 4\sqrt{L/C}$ ) i.e. equation (1) is

(7) $\quad q(t) = e^{-\lambda t} (c_1 \cos(\omega_0\, t) + c_2 \sin(\omega_0\, t)) + A\, G \sin(\omega\, t - \phi)$

where $\lambda = R/(2\, L)$, $\omega_0 = \left(\frac{1}{C\,L} - R^2\right)^{1/2}$, and the gain G and phase shift $\phi$ are given by

(8) $\quad G = \mathrm{Sign}\left(\frac{1}{C\,L} - \omega^2\right) \Big/ \left(L \left(\left(\frac{1}{C\,L} - \omega^2\right)^2 + 4\,\lambda^2\,\omega^2\right)^{1/2}\right)$ and $\phi = \arctan\left(2\,\lambda\,\omega \Big/ \left(\frac{1}{C\,L} - \omega^2\right)\right)$

with the constants $c_1$ and $c_2$ are determined by the initial current and charge.

**Transients vs Steady-State:** The exponentially decaying term in all of the above solutions (determined by the initial conditions) gives the behavior over the short time (determined by the characteristic time $\tau = 1/\lambda$ where $\lambda$ is the appropriate decay constant) of the circuit. The pure trigonometric term $A\, G \sin(\omega\, t - \phi)$ with the appropriate gain G and phase shift $\phi$ is the long term steady state response of the circuit to the applied voltage $a \sin(\omega\, t)$. All our circuits have very short characteristic times an relax to their steady state response very rapidly. An applied voltage that changes sufficiently slowly that the circuit is essentialy always giving a steady state response is called quasi-static. Our chirp signals are quasi static since the frequency changes over a time frame of seconds and the characteristic times for our circuits are much much shorter.

**Quasi-Static Model Predictions:** Neglecting the transient terms and subsituting the particular solutions (with the appropriate gain and phase shift) into the appropriate term(s) in Equation (1,??) gives simple quasi-static expressions for the voltage across and phase shift of that voltage for any component in the circuit as a function of frequency.

**Results:** The predicted voltage gains and phase shifts for the three experiments. Zooming in to examine the phase shifts shows an excellent qualitative match with the phase shifts and reinforces the appropriateness and meaning of the quasi-static approximation. A carefull analysis with sinusoidal fitting of short sections of the experimental response confirms that the quantitative match is excellent.

**Conclusions:** It is possible to use modern audio software and standard computer software to collect high-quality data from electrical circuits. This data shows excellent qualitative and quantitative agreement with standard circuit models in ODE courses. many more interesting models (see for example Sprott [5]) can be constructed and examined using the same tools.

**References:**

[1] Audacity, http://www.Audacity.com, 2008.
[2] Electronics Goldmine, http://www.electronicsgoldmine.com, 2009.
[3] *Mathematica*, http://www.wri.com, 2009.
[4] Polking, Bogess, and Arnold, *Differential Equations*, Prentice Hall, 2001.
[5] Sprott, J. C., American Journal of Physics, volume 68, pp758-763, 2000.