A MORE FAVORABLE ROLE FOR PIECEWISE-DEFINED FUNCTIONS IN CALCULUS

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Abstract: Usually piecewise-defined functions only play the role of a "bad" character in calculus courses by illustrating discontinuity. Modern technology (both with graphing and symbolic algebra) makes it easy to include examples where piecewise-defined functions play a very positive role in the modern world. Calculus techniques do apply to these functions with some care. Topics will include Bezier curves, price functions which offer discounted prices for larger orders, and progressive tax functions. Even the topics of the trapezoid rule and Simpson's rule can be explained in terms of piecewise-defined approximating functions.

Early Practical Examples

UNITED STATES POSTAL SERVICE

The U.S. Postal Service is proposing new rates effective May 11, 2009. The rate for *First-Class Mail Domestic-Retail Letters* will be the following where w represents the weight in ounces and p is the cost in dollars.

$$p(w) = \begin{cases} 0.44 & \text{if} \quad 0 < w \leq 1 \\ 0.61 & \text{if} \quad 1 < w \leq 2 \\ 0.78 & \text{if} \quad 2 < w \leq 3 \\ 0.95 & \text{if} \quad 3 < w \leq 3.5 \end{cases}$$

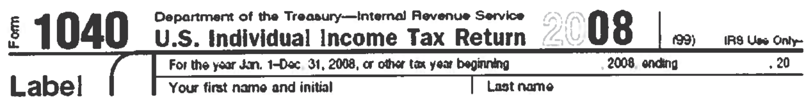
This can be described in words as \$0.44 for not over 1 ounce and \$0.17 for each additional ounce (or fraction thereof) not to exceed 3.5 ounces. Heavier objects should be placed in a *First-Class Mail Domestic-Retail Large Envelope (Flat)* where the rate will be \$0.88 for not over 1 ounce and \$0.17 for each additional ounce (or fraction thereof) not to exceed 13 ounces. *Media Mail* (formerly known as "book rate") will be priced at \$2.38 for not over 1 pound and \$0.39 for each additional pound (or fraction thereof) not to exceed 70 pounds. All of these postal pricing functions are discontinuous, and the "jumps" are all the same with uniform spacing of the subintervals.

A much more irregular piecewise-defined function results if we look at the price of a *First-Class Letter (up to 1 ounce)* over time. This information can be found `http://www.akdart.com/postrate.html` in an unofficial record of the history of the U. S. Postal Service. For example beginning July 1, 1885 such a letter cost 2¢, beginning November 3, 1917 the rate went up to 3¢ (temporarily during World War I), beginning July 1, 1919 the rate went down to 2¢, beginning July 6, 1932 the rate became 3¢ again, and beginning August 1, 1958 it went up to 4¢. In recent years there has been a rate increase nearly every year, particularly since Congress has charged the Postal Service with attempting to cover its costs with postal fees.

Discontinuity is not always a bad thing. Merchants frequently offer a "volume discount" that lowers the price per item when you purchase enough items. Most of the time this pricing is only appropriate for an interger number of items, but you can generalize this to merchandise that is sold by wieght. Suppose that a supermarket charges \$1.39 per pound for pretzels in the bulk food aisle, but they also offer these pretzels at \$1.19 per pound if you purchase at least 3 pounds.

$$p(w) = \begin{cases} \$1.39w & \text{if} \quad 0 < w < 3 \text{ lb} \\ \$1.19w & \text{if} \quad 3 \text{ lb} \leq w \end{cases}$$

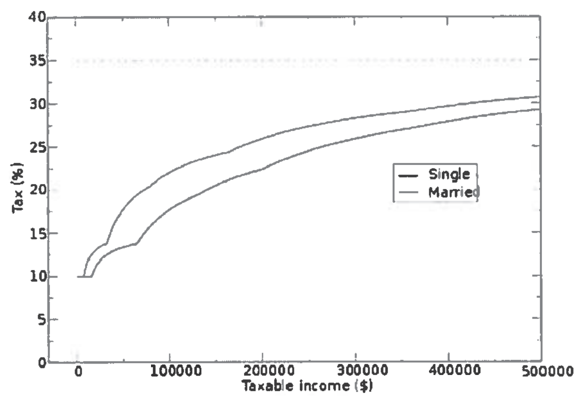
An interesting question here is what weights $w < 3$ lb will it be cheaper to go ahead and buy 3 lb?


Form 1040 Department of the Treasury—Internal Revenue Service U.S. Individual Income Tax Return 2008 (99) IRS Use Only—
Label For the year Jan. 1–Dec. 31, 2008, or other tax year beginning , 2008, ending , 20
Your first name and initial | Last name

Not all piecewise-defined functions are discontinuous. The U.S. Federal Income Tax due in any tax year as a function of your taxable income is a nice example of this (which many people do not understand). For example, the 2008 U.S. Federal Tax F (in dollars) for a person filing as a *Single* with taxable income T (also in dollars) is the following function.

$$F(T) = \begin{cases} 0.10T & \text{if} \quad 0 \le T \le 8{,}025 \\ 802.5 + 0.15(T - 8{,}025) & \text{if} \quad 8{,}025 < T \le 32{,}550 \\ 4{,}481.25 + 0.25(T - 32{,}550) & \text{if} \quad 32{,}550 < T \le 78{,}850 \\ 16{,}056.25 + 0.28(T - 78{,}850) & \text{if} \quad 78{,}850 < T \le 164{,}550 \\ 40{,}052.25 + 0.33(T - 164{,}550) & \text{if} \quad 167{,}550 < T \le 357{,}700 \\ 103{,}791.75 + 0.35(T - 357{,}700) & \text{if} \quad 357{,}700 < T \end{cases}$$

Wikipedia (3-4-09) gives the following graph in an attempt to explain this to a general reader. `http://en.wikipedia.org/wiki/US_Income_tax`



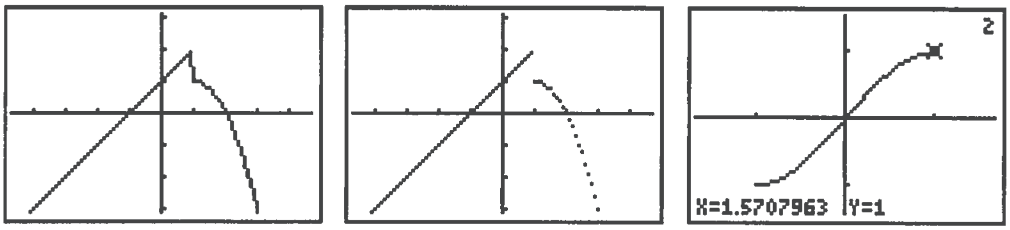
It would be very useful to discuss this tax function F and the above graph in a calculus class, both before the derivative is formally introduced and after. A nice exercise would be to graph F and reproduce the Wikipedia graph on a computer or calculator graphing utility.

Some Examples of Using Technology for Piecewise-Defined Functions

TI-83/84

This family has only a very awkward way to represent most piecewise-defined functions. Here a *test* such $x < 2$ evaluates to the value 0 if the test is false or the value 1 if the test is true. We use this to multiply something we want to "count" only if the test is true. We can also divide by such a test if we want the function to be *undefined* when the test is false (where you would be dividing be zero). Note that you need to use the "Dot Style" if you wish to avoid the *false vertical lines* connecting discontinuous pieces. Thus we can get the following examples of piecewise-defined functions.

$$y_1 = \begin{cases} x+1 & \text{if} \quad x<1 \\ 2x-x^2 & \text{if} \quad x \geq 1 \end{cases} \quad \text{and} \quad y_2 = \begin{cases} \sin(x) & \text{if} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ undefined & \text{otherwise} \end{cases}$$




```
\Y1■(X+1)+(X≥1)(      ·.Y1■(X+1)+(X≥1)(      \Y2■sin(X)/(X≥-π
(2X-X²)-(X+1))         (2X-X²)-(X+1))         /2 and X≤π/2)
```

You can also use the absolute value function to create other common functions (that often are available on other platforms). For example, we can approximate the signum function and the Heaviside function with the following.

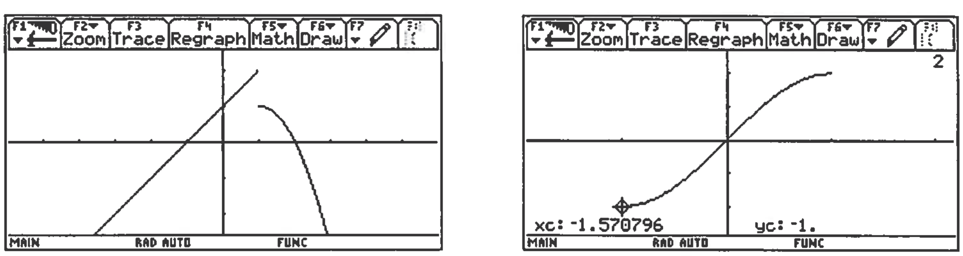
$$\texttt{abs(x)/x} = \begin{cases} -1 & \text{if} \quad x<0 \\ undefined & \text{if} \quad x=0 \\ 1 & \text{if} \quad 0<x \end{cases}$$

$$\texttt{(abs(x)/x+1)/2} = \begin{cases} 0 & \text{if} \quad x<0 \\ undefined & \text{if} \quad x=0 \\ 1 & \text{if} \quad 0<x \end{cases}$$

There are no symbolic operations on this family of graphing calculators. Numerical differentiation and numerical integration work fine with such piecewise-defined functions.

TI-89,Voyage 200, (and very similar commands in TI-Nspire)

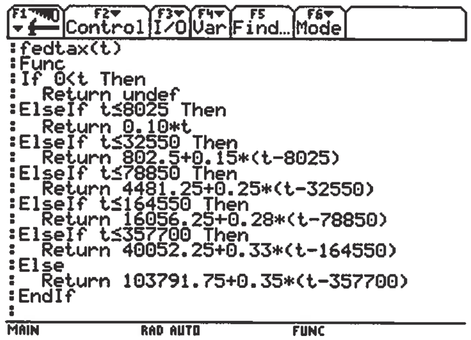
This family has two ways to easily specify piecewise-defined functions. If there are not too many pieces, the `when` command is the simplest method. Here we present y_1 and y_2 from above. Note that in the newer operating systems (here OS 3.10 on a Voyage 200) there is a graphing format option for "Discontiniuty Detection" that will usually avoid the *false vertical lines* at discontinuities.

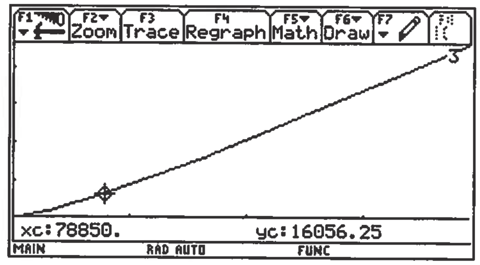


y1(x)=when(x<1,x+1,2*x-x^2) **y2(x)=when(x≥-π/2 and x≤π/2,sin(x),undef)**

$\checkmark y1 = \begin{cases} x+1, x<1 \\ 2\cdot x - x^2, \text{else} \end{cases}$ $\checkmark y2 = \begin{cases} \sin(x), x \geq \frac{-\pi}{2} \text{ and } x \leq \frac{\pi}{2} \\ \text{undef}, \text{else} \end{cases}$

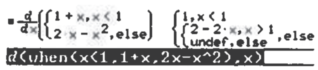
For more than four pieces, it becomes too complicated to use nested `when` commands. It is better to use a function-type program with the `If` – `then` – `elseif` – `else` structure. Here we illustrate by implementing the Single Federal income tax function $F(T)$ described above.

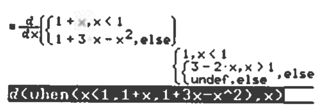


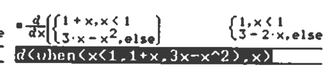


Unfortunately this family of symbolic graphing calculators cannot perform symbolic operations with any function defined in such a function-type program (because there is no way it can interpret what might happen in the control structure you have defined by the commands in the program). Numerical differentiation and numerical integration work fine.

Symbolic differentiation works fairly well within the `when` structure (with the calculator just assuming that the pieces stay the same). It does seem to try to check the existence of the derivative at a *breakpoint*. Thus we get the following.







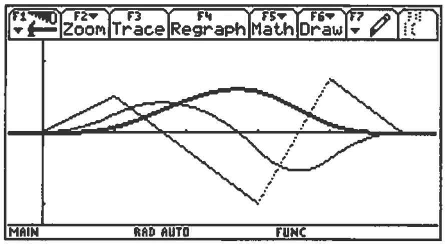
Notice that the second example above actually has no derivative at $x = 1$ because the function is not continuous there. The third example does have a derivative at $x = 1$ and it is correctly noted. Apparently if there is an *undef* interval (as with the restricted sine function above), it is not able to find a symbolic derivative.

While you would not want to present alot of theoretical information about *B-splines* and their importance in numerical analysis to students in calculus, there is nothing wrong with looking at specific examples. Here is one.

$$b(x) = \begin{cases} \frac{1}{12}x_3 & \text{if} \quad 0 \le x < 1 \\ \frac{1}{24}\left(-3x^3 + 15x^2 - 15x + 5\right) & \text{if} \quad 1 \le x < 3 \\ \frac{1}{24}\left(7x^3 - 75x^2 + 255x - 265\right) & \text{if} \quad 3 \le x < 4 \\ \frac{1}{8}(5-x)^3 & \text{if} \quad 4 \le x < 5 \\ 0 & \textit{otherwise} \end{cases}$$

This piecewise cubic has a continuous first and second derivative everywhere. Even though it is complicated, we implement it with nested `when` statements so that we can find symbolic derivatives. (For some reason it has trouble with the derivative at $x = 0$.) The plot below includes b (in Thick), b' and b''.

```
Define b(x) = when(x < 3, when(x < 1, when(x < 0, 0, x^3/12),
    (-3x^3 + 15x^2 - 15x + 50/24), when(x < 5, when(x < 4,
    (7x^3 - 75x^2 + 255x - 265)/24, (5 - x)^3/8), 0))
```



$$\blacksquare\ \text{Define } b(x)=\begin{cases}\begin{cases}\begin{cases}0, & x<0\\ \frac{x^3}{12}, & \text{else}\end{cases} & ,x<1\\ \frac{-3\cdot x^3+15\cdot x^2-15\cdot x+5}{24}, & \text{else}\end{cases} & ,x<3\\ \begin{cases}\frac{7\cdot x^3-75\cdot x^2+255\cdot x-265}{24}, & x<4\\ \frac{(5-x)^3}{8}, & \text{else}\end{cases} & ,x<5\\ 0, & \text{else}\end{cases}$$

$$\begin{cases}\begin{cases}\begin{cases}0, & x<0\\ \begin{cases}\frac{x^2}{4}, & x>0\\ \text{undef}, & \text{else}\end{cases} & ,\text{else}\end{cases} & ,x<1\\ \frac{-(3\cdot x^2-10\cdot x+5)}{8}, & \text{else}\end{cases} & ,x<3\\ \begin{cases}\frac{7\cdot x^2-50\cdot x+85}{8}, & x<4\\ \frac{-3\cdot(x-5)^2}{8}, & \text{else}\end{cases} & ,x<5\\ 0, & \text{else}\end{cases}$$

d(b(x),x)

$$\begin{cases}\begin{cases}\begin{cases}0, & x<0\\ \begin{cases}\begin{cases}\frac{x}{2}, & x>0\\ \text{undef}, & \text{else}\end{cases} & ,x>0\\ \text{undef}, & \text{else}\end{cases} & ,\text{else}\end{cases} & ,x<1\\ \frac{-(3\cdot x-5)}{4}, & \text{else}\end{cases} & ,x<3\\ \begin{cases}\frac{7\cdot x-25}{4}, & x<4\\ \frac{-3\cdot(x-5)}{4}, & \text{else}\end{cases} & ,x<5\\ 0, & \text{else}\end{cases}$$

d(ans(1),x)

In addition to the absolute value function, the TI-89 and Voyage 200 have the following piecewise defined functions: `floor`, `ceiling`, `sign`. We can get the Heaviside function and the truncated power function easily from these.

$$\texttt{heavi(x)} = (\texttt{sign(x)}+1)/2 = \begin{cases} 0 & \text{if} \quad x<0 \\ (sign(0) & \text{if} \quad x=0 \\ 1 & \text{if} \quad 0<x \end{cases}$$

$$\texttt{tpower(x)} = (\texttt{abs(x)}+\texttt{x})/2 = (x)_+ = \begin{cases} 0 & \text{if} \quad x<0 \\ x & \text{if} \quad 0\le x \end{cases}$$

You can create piecewise-defined functions using these functions instead of using the `when` command or using a function-type program. The calculator will be better able to do symbolic integration of the type $\int_0^x f(t)\,dt$ using these expressions because it understands the "ordering" of the pieces.

Maple

This computer algebra system has many important piecewise-defined functions: `abs`, `ceil`, `floor`, `sign`, `signum`, and `Heaviside`. However for most of the simple examples in calculus, the command `piecewise` will work nicely. Here are some examples.

$$\texttt{y1 := x->piecewise(x<1,x+1,2*x-x^2)}$$

$$\text{gives} \begin{cases} x+1 & \text{if} \quad x<1 \\ 2x-x^2 & \text{if} \quad 1\le x \end{cases}$$

$$\texttt{f := x->piecewise(-Pi/2<=x and x<=Pi/2,sin(x))}$$

$$\text{gives} \begin{cases} \sin(x) & \text{if} \quad -\frac{\pi}{2}\le x\le\frac{\pi}{2} \\ 0 & otherwise \end{cases}$$

$$\texttt{g := x->piecewise(-Pi/2<=x and x<=Pi/2,sin(x),undefined)}$$

$$\text{gives} \begin{cases} \sin(x) & \text{if} \quad -\frac{\pi}{2}\le x\le\frac{\pi}{2} \\ undefined & otherwise \end{cases}$$

For plotting, the *false vertical lines* at the discontinuities can be avoided by adding the optional argument `discont = true`.

```
> plot(piecewise(x < 1,x+1,2*x-x^2),x = -1..3,discont = true);
```

There is even a `convert/piecewise` command to convert other functions into this format, and a `convert/Heaviside` command to rewrite things in terms of the Heaviside function (although the documentation contains the warning that the expression may change value at a finite number of points – probably meaning some breakpoints may get switched to the adjacent piece or become undefined).

> $b2 := \frac{\qquad}{24};$

$$b2 := -\frac{1}{8}x^3 + \frac{5}{8}x^2 - \frac{5}{8}x + \frac{5}{24}$$

> $b3 := \frac{(7\cdot x^3 - 75\cdot x^2 + 255\cdot x - 265)}{24};$

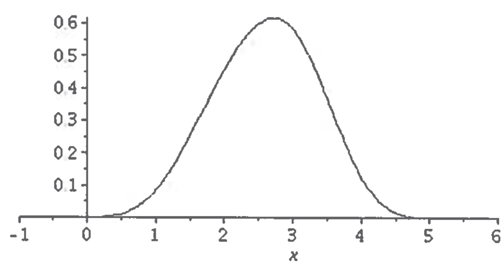
$$b3 := \frac{7}{24}x^3 - \frac{25}{8}x^2 + \frac{85}{8}x - \frac{265}{24}$$

> $b4 := \frac{(5-x)^3}{8};$

$$b4 := \frac{1}{8}(5-x)^3$$

> $b := \mathit{piecewise}(x < 0, 0, x < 1, b1, x < 3, b2, x < 4, b3, x < 5, b4, 0);$

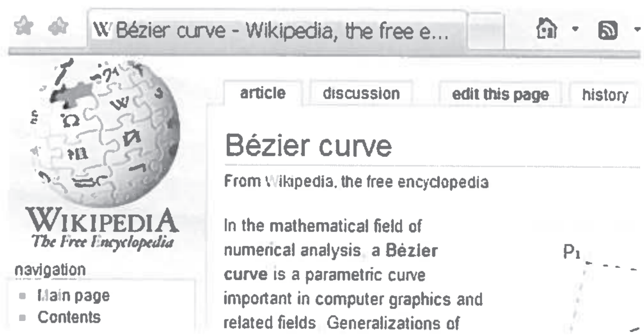
$$b := \begin{cases} 0 & x < 0 \\ \frac{1}{12}x^3 & x < 1 \\ -\frac{1}{8}x^3 + \frac{5}{8}x^2 - \frac{5}{8}x + \frac{5}{24} & x < 3 \\ \frac{7}{24}x^3 - \frac{25}{8}x^2 + \frac{85}{8}x - \frac{265}{24} & x < 4 \\ \frac{1}{8}(5-x)^3 & x < 5 \\ 0 & \mathit{otherwise} \end{cases}$$



Bezier Curves

It is also true that few calculus textbooks have any practical, realistic examples of *parametric equations* (or equivalently *vector-valued functions*). Without going into the details needed in Computer-Aided Geometric Design courses, it is easy to introduce the simple concept of a cubic Bezier curve and have calculus students explore some of the properties. Here we look at only one piece, but all of the important applications (including representing the outlines of nearly all fonts on laser printers) use several pieces and fit into the *piecewise-defined theme* of the this paper.

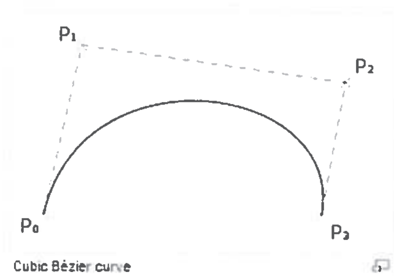
I would highly recommend the Wikipedia article on Bezier curves. `http://en.wikipedia.org` It nicely introduces the concept and points to more advanced references. The article has Java animations, and many other illustrations.



Briefly we define a single piece of a Bezier curve. Suppose we have four points in the plane (called *control points*) $P_0 = (p_0, q_0)$, $P_1 = (p_1, q_1)$, $P_2 = (p_2, q_2)$, and $P_3 = (p_3, q_3)$. Then the piece specifed by these four points is the represented by the following.

$$\begin{aligned} x &= p_0(1-t)^3 + p_1\left(3(1-t)^2 t\right) + p_2\left(3(1-t)t^2\right) + p_3 t^3 \\ y &= q_0(1-t)^3 + q_1\left(3(1-t)^2 t\right) + q_2\left(3(1-t)t^2\right) + q_3 t^3 \\ & \qquad 0 \le t \le 1 \end{aligned}$$

Here is a plot of a typical piece (from the Wikipedia article).



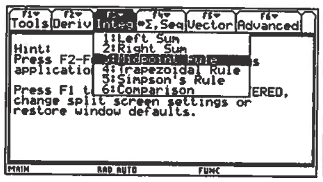
Cubic Bézier curve

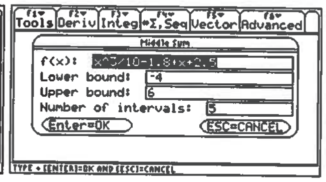
I always give my students in calculus the assignment to design a letter (in script style with no straight lines) using a few Bezier curve pieces. Usually they pick one of their initials. They can have this plotted in a calculator or on a computer as they work out the design, but what they hand in is the set of control points. Then I plot each one, and add it to a class report where they can see their own design and those of all of the others in the class. I wrote calculator programs for the HP-28S, Sharp EL-5200, and CASIO fx-7000 to plot out Bezier curves nearly 20 years ago, and I have been able to easily get these curves on every graphing calculator that has come out since. My students always like this topic very much. They learn to move the control points to adjust the slope of the parametric curve in order to get the shape they desire to make a part of their letter.

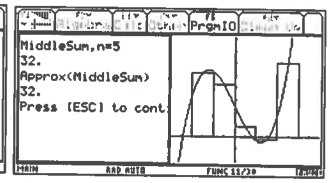
Trapzoidal and Simpson's Rule

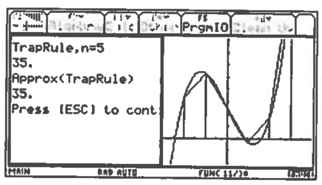
If you work more with piecewise-defined functions earlier, then when you cover numerical integration you can simply mention that almost all of the methods used correspond to approximating the integrand f with some type of piecewise-defined function g. You simply integrate the approximating function g (which you can easily do) instead of the more difficult f. For example, the *midpoint rule* approximates f with a carefully chosen piecewise constant function (which is not continuous). The *trapezoidal rule* approximates f with continuous piecewise linear function. *Simpson's rule* approximates f with a continuous piecewise quadratic function. However this only works if students understand how to integrate a piecewise-defined function. In particular, if a function is not continuous (or not continuously differentiable), you need to apply the Fundamental Theorem of Calculus carefully. Usually this involves breaking up the definite integral into smaller integrals on the intervals defining the pieces.

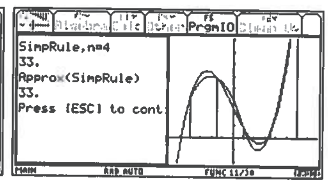
Here we show some simple examples using the display for numerical integration provided on the Voyage 200 Application called *Calc Tools*.











Notice that even though Simpson's rule provides the exact answer for a cubic polynomial integrand f in the example, the piecewise defined quadratic approximating function g in the last screen shot does not exactly approximate the original function everywhere (these two just have the same integral over this interval).