USING MATHEMATICA® TO VISUALIZE PARTIAL DIFFERENTIAL EQUATIONS

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Introduction

Mathematica® can be used to help students to visualize some of the important concepts in an introductory course in partial differential equations. While teaching such a course, I developed several demonstrations to illustrate the convergence of Fourier series, vibrating strings and membranes, heat flow, and the hanging chain problem.

The Vibrating String and the Method of D’Alembert

Consider a string with constant linear density that is stretched between two fixed points \( x = 0 \) and \( x = L \) on the \( x \)-axis. Let \( u = u(x,t) \) be the transverse displacement of the string at \( x \) \((0 < x < L)\) at time \( t \). It can be shown that \( u = u(x,t) \) must satisfy the one-dimensional wave equation with boundary and initial conditions:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \text{ where } 0 < x < L \text{ and } t > 0,
\]

\[
u(0,t) = 0 \text{ and } u(L,t) = 0 \text{ for } t > 0, \text{ and}
\]

\[
u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x), \text{ for } 0 < x < L.
\]

Here \( c^2 = \frac{T}{\rho} \) (\( T \) is the tension in the string and \( \rho \) is the linear density of the string) and the \( f \) and \( g \) are given functions that describe the initial position and initial velocity of the string.

There are two ways we will solve this problem: (1) the standard separation of variables and express the solution as a Fourier sine series and (2) d’Alembert’s solution that expresses the solution in terms of traveling waves:

\[
u(x,t) = \frac{1}{2} [f^-(x-ct) + f^+(x+ct)] + \frac{1}{2c} \int_{-\infty}^{\infty} g^*(s) ds,
\]

where \( f^- \) and \( g^* \) are the odd periodic extensions of \( f \) and \( g \).

227
Take \( L = 1, \quad c = 1, \) and \( g(x) = 0 \) for \( 0 \leq x \leq L = 1. \) It is easier to see the traveling waves if the function \( f \) is 0 for most of the unit interval. The function \( f \) will be the piecewise linear function given by
\[
f(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{1}{4} \text{ or } \frac{1}{2} \leq x \leq 1 \\
16x - 4 & \text{if } \frac{1}{4} \leq x \leq \frac{5}{16} \\
-8x + \frac{7}{2} & \text{if } \frac{5}{16} \leq x \leq \frac{3}{8} \\
4x - 1 & \text{if } \frac{3}{8} \leq x \leq \frac{7}{16} \\
-12x + 6 & \text{if } \frac{7}{16} \leq x \leq \frac{1}{2}
\end{cases}
\]

In Mathematica\textsuperscript{\textregistered}, this piecewise-defined function can be entered as follows. The graph is given in Figure 1.

\[
f(x) := \text{If}[\frac{1}{4} \leq x \leq \frac{1}{2}, 1, 0] \cdot \text{If}[\frac{1}{4} \leq x \leq \frac{3}{8}, 16x - 4, -8x + \frac{7}{2}, \text{Min}[16x - 4, -8x + \frac{7}{2}], \text{Min}[4x - 1, -12x + 6]]
\]

![Graph of f](image)

Figure 1 – The graph of \( f \), the initial position of the string.

We now calculate the coefficients of the Fourier sine series for \( f \).

\[
a_n = 2 \int_0^1 f(x) \sin n\pi x \, dx
\]

\[
= 2 \left( \int_{1/2}^{5/16} (16x - 4) \sin n\pi x \, dx + \int_{5/16}^{3/8} (-8x + \frac{7}{2}) \sin n\pi x \, dx + \int_{3/8}^{7/16} (4x - 1) \sin n\pi x \, dx + \int_{7/16}^{1} (-12x + 6) \sin n\pi x \, dx \right)
\]

After evaluating and simplifying, we obtain

\[
a_n = \frac{-8}{\pi^2 n^2} \left( 4 \sin \frac{\pi n}{4} - 6 \sin \frac{5\pi n}{16} + 3 \sin \frac{3\pi n}{8} - 4 \sin \frac{7\pi n}{16} + 3 \sin \frac{\pi n}{2} \right)
\]

The \( n \)th partial sum of the Fourier sine series is \( \sum_{k=1}^{n} a_k \sin k\pi x \). The graphs of the partial sums of the Fourier sine series superimposed on the graph of \( f \) is given in Figure 2.
Figure 2 – Graphs of partial sums of the Fourier sine series of \( f \).

We see that taking 20 terms of the series gives a good visual approximation to the initial position of the string. The solution of the problem will be \( u(x,t) = \sum_{k=1}^{\infty} a_k \sin k \pi x \cos k \pi t \), use the first 20 terms for the approximate solution to graph. An animation can be shown using Mathematica® by using the command:

\[
\text{Do[Plot[Evaluate[} \sum_{k=1}^{20} a[k] \sin[k \pi x] \cos[k \pi t], \{x, 0, 1\}, \text{PlotRange -> (-1, 1)}]\}, \{t, 0, 2, .05\}]
\]

Figure 3 shows the displacement at \( t = 0.0, 0.2, 0.4, 0.6, 0.8, \) and 1.0.

Figure 3 – The displacement of the string.

By looking at the d’Alembert method we get a new insight into the behavior of the vibration of the string. For our problem, the d’Alembert method gives the solution in the form \( u(x,t) = \frac{1}{2} \left[ f^−(x−t) + f^+(x+t) \right] \), where \( f^+ \) denotes the odd periodic extension of \( f \). The first term, \( f^−(x−t) \), represents this extension of \( f \) moving to the right and the second term, \( f^+(x+t) \), represents this extension of \( f \) moving to the left. Figure 4 shows the graph of \( u(x,t) = \frac{1}{2} \left[ f^−(x−t) + f^+(x+t) \right] \) for \( t = 0.0, 0.2, 0.4, 0.6, 0.8, \) and 1.0.
The Hanging Chain Problem

A chain with uniform density is hanging from a support. The $x$-axis is vertical and $x = 0$ at the bottom of the chain; $x = L$ at the top. The $u$-axis is horizontal and the transverse movements of the chain are in the $xu$-plane. The differential equation is given by

$$\frac{\partial^2 u}{\partial t^2} = g \left( x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \right),$$

$$u(L,t) = 0, \text{ for } t > 0, \text{ and}$$

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = h(x), \text{ for } 0 < x < L,$$

where $g$ is the gravitational acceleration and $f$ and $h$ are given functions. Take $g = L = 1,$ $h(x) = 0, \quad f(x) = \begin{cases} 0.01 & \text{if } 0 \leq x \leq 0.5 \\ 0.02 (1 - x) & \text{if } 0.5 \leq x \leq 1 \end{cases}.$ By the separation of variables method, we see that the solution will be a Bessel series. The solution of the problem is given by the series:

$$u(x,t) = \sum_{j=1}^{\infty} \frac{0.04 \left(2J_2(\alpha_j) - J_2(\alpha_j \sqrt{5})\right)}{\left(\alpha_j J_1(\alpha_j)\right)^2} J_0(\alpha_j \sqrt{x}) \cos \left(\frac{\alpha_j t}{2}\right),$$

where $J_0, J_1, \text{ and } J_2$ are the Bessel functions of order 0, 1, and 2, respectively, and $\alpha_j$ is the $j^{th}$ positive zero of the Bessel function of order 0. Use Mathematica® to obtain these Bessel coefficients and the solution (for $t = 0$).
\[ u(x, t) = \sum_{j=1}^{15} \left( \frac{0.04}{(\alpha(\{[j]\}) \text{BesselJ}[1, \alpha(\{[j]\}))^2 (2 \text{BesselJ}[2, \alpha(\{[j]\}]) - \text{BesselJ}[2, \alpha(\{[j]\}) \sqrt{0.5}] \right) \cdot \text{BesselJ}[0, \alpha(\{[j]\}) \sqrt{x}] \cos\left(\frac{\alpha(\{[j]\})}{2} t\right) \right) \]

Do[ParametricPlot[{u[x, t], x}, {x, 0, 1}, PlotRange -> {(-0.01, 0.01), (0, 1)}, Ticks -> {(-0.01, 0.01), (0, 0.5, 1)}, PlotStyle -> {Hue[1]}], {t, 0, 6, 0.1}]

Figure 5 shows the position of the chain at \( t = 0.0, 0.5, 1.0, 1.5, 2.0, \) and 2.5.

![Figure 5 – The position of the chain.](image)

In conclusion, the study of the partial differential equations of mathematical physics offers a rich environment for the use of the Mathematica® to show the connection between the mathematics and the physical model.

References:
