

Demos with Positive Impact: A Tool in the Student-Centered Classroom

David R. Hill
Mathematics Department
Temple University
Philadelphia, PA 19122
hill@math.temple.edu

Lila F. Roberts
Mathematics and CS Department
Georgia Southern University
Statesboro, GA 30460
lroberts@gasou.edu

Introduction

As instructors, we use a variety of tools to make mathematical concepts real and meaningful to our students. Even in laboratory classes or classroom environments in which cooperative learning activities are prevalent, the instructor plays an important role as facilitator of learning. [Demos with Positive Impact](#) is an NSF project to develop a web-based collection of effective instructional demonstrations and to connect this resource to mathematics instructors. In this article, we describe the project and showcase some of the demos that are currently in the [Demos with Positive Impact](#) collection. In addition, we provide information on how instructors participate in the project as contributors and end-users.

Demos in the Mathematics Classroom

As a **demo**, we have in mind a vignette incorporated within a classroom presentation that engages the learner on a level in addition to the dialog of the instructor. Demos provide a description or explanation of an idea, concept, or process, illustrated with some form of instructional technology such as computer software for computation or visualization, graphing calculator, physical object or prop, or data collection apparatus. In contrast to lab activities or projects, these vignettes are designed to be presented by the instructor. Demos have an important place in any classroom environment: lecture, lab, project-centered, or collaborative learning.

Every experienced mathematics instructor has a private toolbox of demos that have been effective in the classroom. [Demos with Positive Impact](#), funded by an NSF proof-of-concept grant, focuses on collecting, developing, and disseminating these valuable tools to our colleagues across the country. The scope of the project is broad, encompassing topics from college preparatory mathematics to post-calculus mathematics. [Demos with Positive Impact](#) is a resource for instructors who are looking for ideas or demos that are adaptable for various teaching styles and learning environments; it is useful for high school and college mathematics instructors.

The success of [Demos with Positive Impact](#) depends on tapping into a largely unharvested resource: our colleagues' collective experiences in mathematics

teaching. The web site contains a growing collection of demos that have primarily been designed from ideas submitted to us by various colleagues. The remainder of this article focuses on showcasing several of the clever ideas that can make an important impact in teaching mathematics in the undergraduate mathematics curriculum.

Volumes of Solids of Revolution

One of the main applications of definite integrals presented in a Calculus course involves solids of revolution. As with the general area problem, volumes are developed via an approximation process that leads, in the limiting case, to a definite integral. Often students tend to focus on the final “formula,” and have blind faith in their ability to memorize and remember the details of the formula. Rather than a presenting formula-driven approach, we find it very useful (and helpful) to develop the ideas of how the approximations come about. We do this by using a variety of props and visualization tools. The demos build a toolbox of teaching aids to increase student understanding. Here we consider two approaches to computing volumes: disks and shells.

The Method of Disks—Props and Visualization Tools

A common paper wedding bell decoration (Figure 1) nicely illustrates a solid of revolution. It is easily transported to the classroom and can be used both by the instructor and the student.

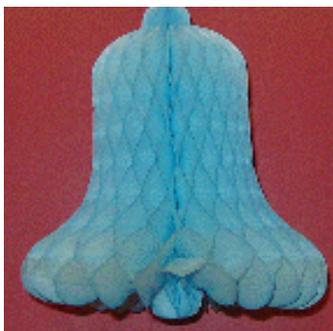


Figure 1. Wedding Bell Decoration

We can illustrate the generation of a solid of revolution about a horizontal axis. Here we held the bell against a vertical board and let it open by itself. The paper folds uncompress illustrating the rotation of the planar region shown in Figure 2 into the 'solid bell'.

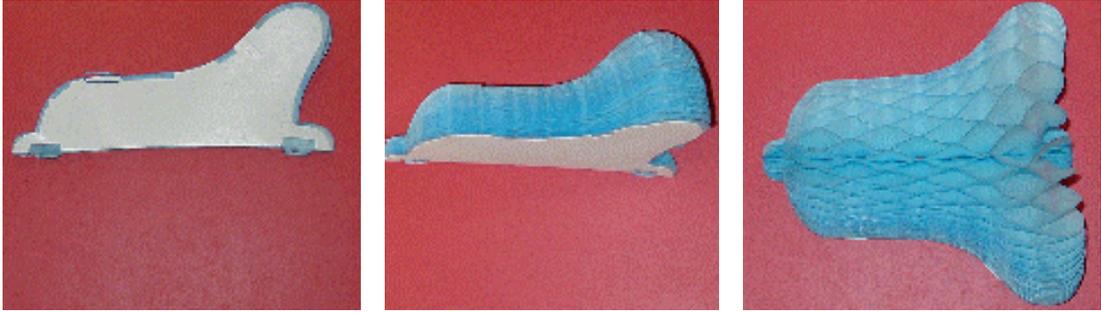


Figure 2. Generating the “Solid” Bell

The produce section of your favorite market is a good place to find props that nicely illustrate solids of revolution. To see this, consider two of the types of regions we often revolve about the x -axis shown in Figure 3 and cucumber and squash props that illustrate the solids.

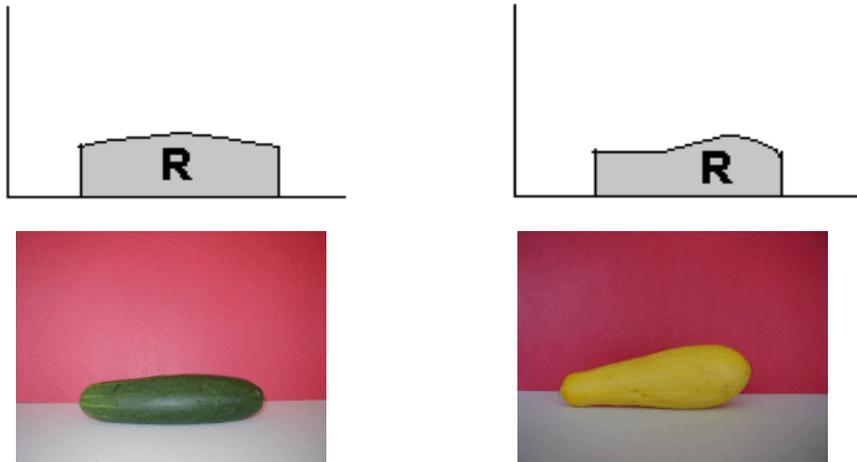


Figure 3. Vegetable Props

For the regions R in this demo we “slice” the planar region into strips of area and revolve the strips around the x -axis to generate “disks”. The disks are cylinders that approximate the volume of the actual slices of the solid generated by the revolution of region R .

In Figure 4, a curve and one strip of area, which generates the cylindrical disk, are shown. The circular faces of the cylindrical disk are called the cross sections of the solid. In addition, the “cross sections” of the cucumber prop are shown.

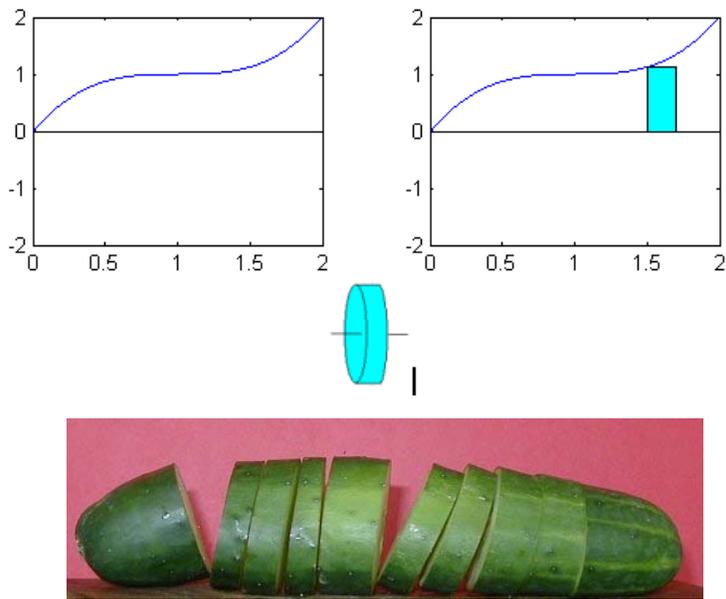


Figure 4. Cross Sections that Approximate the Volume

To allow for interactive visualization of the disk method, MATLAB and *Mathematica* routines are provided. Figure 5 shows the approximating disks for a “wedding bell” function as generated by MATLAB.

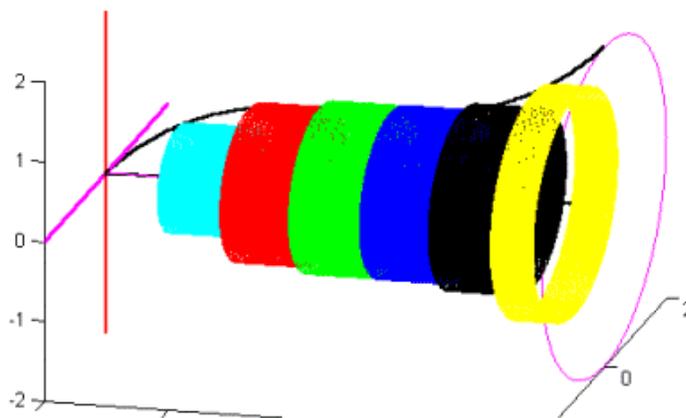


Figure 5. Wedding Bell Disks. To see an animation, click [HERE](#).

In addition to the MATLAB and *Mathematica* routines, a [gallery of animations](#) has been developed that can be played within a browser (animated gifs), Windows Media Player (avi), or Quicktime (mov). These animations and movies are nice examples to bring into the classroom.

The Method of Shells—More Props and Visualizations

The disk method involves slicing the solid into cross sections that are cylindrical disks. An alternate approach to volume calculations involves filling the solid with nested cylindrical shells. Students frequently have trouble developing a helpful mental image to accompany these concepts.

Another trip to the produce market provides yet another useful prop. The idea of “nesting” or “layering” can be nicely illustrated using an onion, shown in Figure 6.



Figure 6. The Layers of an Onion

Alternatively is a set of Matroyska **dolls** (Figure 7) can be used as an effective prop. The hollow dolls of varying sizes nest together compactly.



Figure 7. Matroyska Dolls

Once students understand how the approximating elements will fit together, we need to show what kind of approximating element we will use and how to compute its volume. The snack crackers, Combos, shown in Figure 8, are excellent props to illustrate the cylindrical shells. Sharing these crackers with students is a fun way to help them develop the visualization skills that can help give meaning to the approximation scheme.



Figure 8. Illustrating Shells using Combos

Finally, to motivate the technique for computing the volume of the shell, a cut and flattened paper towel or toilet paper roll (Figure 9) is very instructive.



Figure 9. Cut and Flattened Toilet Paper Roll

We can think of a shell as being "generated" by a rectangle with height $f(x)$ and width Δx . If the graph of $y = f(x)$ is revolved about the y -axis, the radius of a shell is measured from the axis of revolution, given by x . The height of a shell will have height $f(x)$, and the thickness is Δx .

In Figure 10, we show a "half-shell" constructed within a solid of revolution. Note how the half-shell looks like our Combo prop.

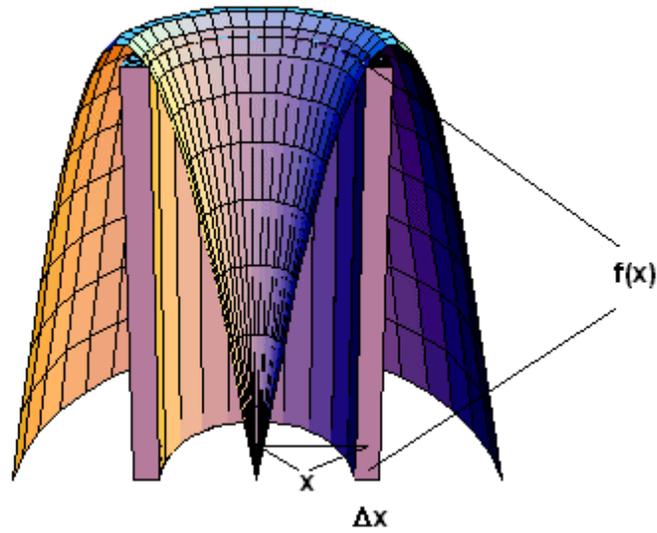


Figure 10. A Half-shell Constructed within a Solid of Revolution

A *Mathematica* notebook has been developed to generate frames that can be played as an animation of the approximation process. Half-shells are generated within the solid so that the nesting procedure is easy to visualize. A solid with ten approximating shells are shown in Figure 11.

Additionally, a MATLAB M-file has been developed that illustrates several steps of an approximating scheme using the method of shells. Shells are constructed and displayed within an [animation](#).

Half-Surface and Half-Shells

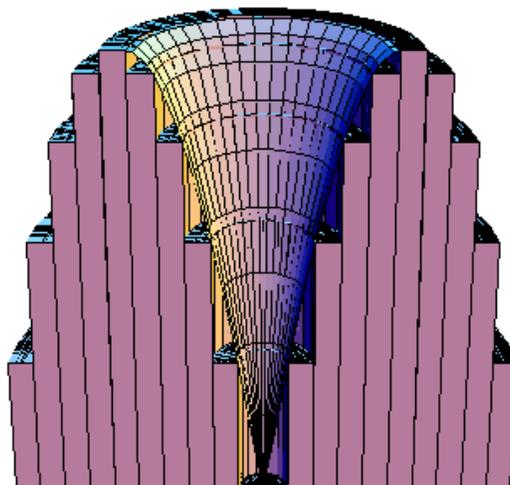


Figure 11. A Solid and Approximating Shells

Monte Carlo Methods—Bacteria and Rainbows

Monte Carlo simulation is a technique for approximating a desired probability without performing the actual experiment. This type of mathematical model teaches students how to represent a real-world system in terms of mathematical relations. Because the technique is commonly used to solve actual problems and is conceptually easy to introduce, it provides a good introduction to probability. In fact, there is no need to try and explain the general concept a priori, the experience students get from a few experiments often lays sufficient foundation for them to see how the technique applies in a variety of situations.

[Demos with Positive Impact](#) has a collection of Monte Carlo simulations for various physical situations. Two such demos deal with Bacterial Allocation and generating Rainbows. Java applets to perform the Monte Carlo simulation have been constructed to accompany the demos.

[Bacteria Allocation](#)

This demo uses the Monte Carlo technique to simulate a count of bacteria that are present as a result of a certain sampling process. The process extracts a fixed number of droplets **N** from a sample of contaminated liquid in which bacteria are randomly distributed. It is observed that (on the average) only a certain percent **K** of the droplets contain the bacteria. We want to approximate (on the average) the number of bacteria that are present in the droplets.

A scientist collects **N** droplets from a sample of a liquid containing bacteria. When the scientist studied the droplets, she found that only **K%** of the **N** droplets collected contained bacteria. This process was repeated a number of times and it was found that on the average only **K%** of the droplets were contaminated with bacteria, but the total number of bacteria from those droplets varied. To approximate the average number of bacteria contained in the contaminated droplets of this particular liquid the following simulation was devised.

For experimental purposes set **N** = 100 and **K** = 50. Imagine that the 100 droplets are arranged in a set of containers that form a square, 10 containers by 10 containers. Each of the containers is assigned a number from 1 to 100. (See the numbers in black in the lower left corner of the containers in Figure 12.) We randomly generate integers from 1 to 100. When a particular container's number is generated we add 1 "bacteria" to the container. We repeat this process until 50 containers contain at least one bacterium and we keep a running total of the number of bacteria that are allocated to the containers. In Figure 10 below we show a result of this simulation. In this particular experiment 73 bacteria were needed to have 50 of the 100 droplets contain at least one bacterium.

0	3	0	1	0	0	0	1	0	0
10	20	30	40	50	60	70	80	90	100
3	2	2	0	1	0	0	3	2	0
9	19	29	39	49	59	69	79	89	99
1	0	1	0	2	1	0	2	0	1
8	18	28	38	48	58	68	78	88	98
2	0	1	1	1	1	1	0	0	0
7	17	27	37	47	57	67	77	87	97
0	1	0	0	0	1	0	1	1	0
6	16	26	36	46	56	66	76	86	96
2	0	0	0	0	1	0	1	0	1
5	15	25	35	45	55	65	75	85	95
0	1	2	1	0	0	0	0	0	0
4	14	24	34	44	54	64	74	84	94
1	2	0	1	1	2	0	0	0	1
3	13	23	33	43	53	63	73	83	93
2	0	4	0	1	0	0	1	1	2
2	12	22	32	42	52	62	72	82	92
1	0	1	0	1	2	2	1	0	0
1	11	21	31	41	51	61	71	81	91

Figure 12. Simulation of Bacteria Allocation. To see an animation, click [HERE](#).

By repeating the procedure a large number of times with $N = 100$ and $K = 50$ we can obtain a good estimate of the average number of bacteria that occur in the 50 droplets. A MATLAB routine implements the strategy outlined above. Additionally, a [Java applet](#) having the same functionality has been developed.

[Making Rainbows](#)

We have been fascinated by the phenomena of a rainbow throughout time. It was Sir Isaac Newton's experiment with light passing through a prism that revealed a band of colors varying from red to violet. The colors of the band are violet, indigo, blue, green, yellow, orange, and red. (A mnemonic that is often used to recall these colors is **VIBGYOR** or Richard Of York Gained Battle In Vain. An alternate form, ROY G. BIV is also used.)

Newton concluded that white light is really a mixture of colored lights, and that each color bends differently as it passes through a prism. Thus we see the band of colored light, which is called the *visible spectrum*. The light we can see is but a portion of the huge spectrum of energy called electromagnetic radiation.

A rainbow is formed by light from the sun hitting a raindrop. When this occurs the light is **refracted** (the change of direction of light in passing from one medium to another) into the colors of the spectrum and then **reflected** (the return of the light from striking a surface) off the back of the raindrop. Each color of the spectrum is refracted at a slightly different angle depending upon its wavelength. For instance violet light is refracted more than red light. As the spectrum of light emerges from the drop an observer on the ground will see only one color depending upon the angle of observation.

The rainbow we normally see is called the **primary rainbow** and is produced by one internal reflection within the drop. When conditions are just right a fainter larger **secondary rainbow** with colors in reverse order appears. This is the result of rays undergoing a second reflection within the drop.

The velocity of light is dependent upon the medium through which it passes, like air or water. The change in velocity of light as it passes from air to water cause the refraction, bending, described above. The **refraction index** or **index of refraction** is the ratio of the velocities of light in the two media. We can use the refraction indices of the colored light to set up the appropriate angles of refraction and reflection in our simulation. This information together with algebra and trigonometry can be blended to model the color dispersion we see in the phenomena of a rainbow.

To develop the model, imagine a large number of parallel rays hitting a spherical rain drop. For each ray we plot a single point of light in accordance with the laws of reflection and refraction. For each ray of light we randomly choose a color that will be seen by an observer once it has been refracted and reflected either resulting in a point of light in the primary or secondary rainbow. Since we are randomly assigning a color from the spectrum to a ray, the process is using a Monte Carlo simulation. In order to get the angles correct we use the refraction index of the colored light for the mediums air and water. This also determines the placement of the corresponding dot of light in our picture. Figure 13a shows a result of a simulation using 15,000 rays. *In this simulation we have used only the three primary colors red, green and blue for simplicity.* Even with this small number of rays the primary and secondary rainbows are evident, with of course a lot of 'scatter' within the primary arc. In Figure 13b the rainbow is simulated using 30,000 rays. The result is a stronger definition of both the primary and secondary arcs.

Figure 13 was generated using a MATLAB M-file for the Monte Carlo simulation. A [Java applet](#) with the same functionality has been developed.

Demos with Positive Impact: An Invitation

Every experienced mathematics instructor has his or her own private toolbox of instructional demonstrations that have been useful for teaching mathematical concepts.

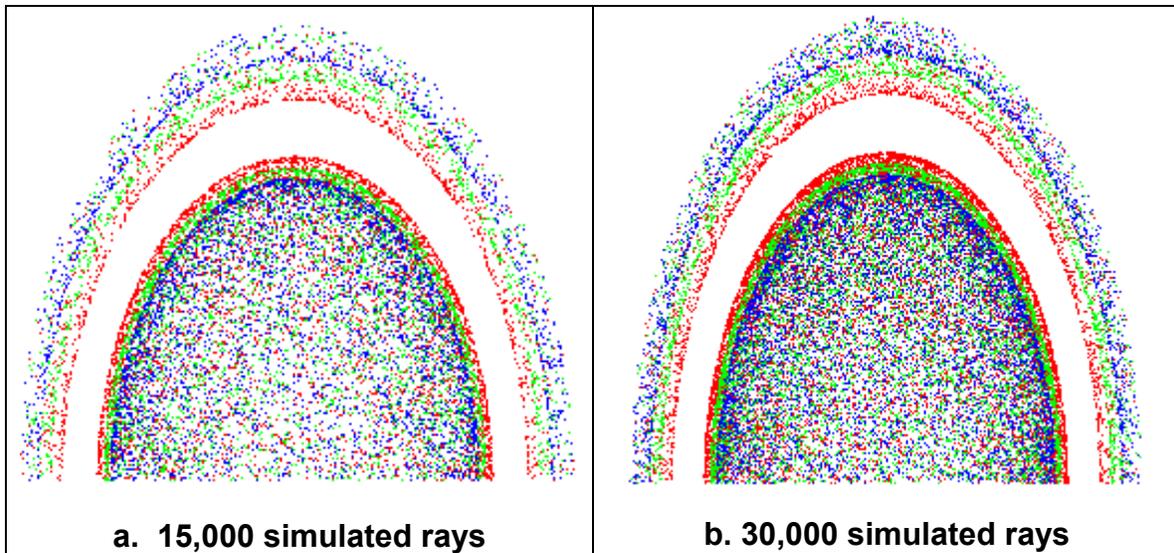


Figure 13. MATLAB Simulation of a Rainbow

The [Demos with Positive Impact](#) project recognizes that faculty members are often not encouraged nor rewarded for activities related solely to development of pedagogical tools, so while we cannot pay contributors for their ideas, we can provide a mechanism by which mathematics instructors can get some recognition for innovative teaching strategies. We invite you to visit the growing [Demos with Positive Impact](#) web site for an on-line opportunity to contribute to the project. In addition, we welcome feedback on the collection or on individual demos within the collection.

GOT DEMOS?

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