

■ Example 9.

Find a power series solution of the following initial value problem:

$$x''(t)+4x(t)=0; x(0)=1 \text{ and } x'(0)=0.$$

The coefficients of this equation are analytic everywhere and in particular at 0. Thus we expect to find a power series solution of the form: $x[a,t,k]$. Our task is to find these coefficients. Denote them by $a[n]$. Let $x[a,t,k]$ denote the k -th partial sum of the power series solution. Substitute it into the given equation and collect the like powers of t . Below, it is done for $k=8$.

```
x[a_,t_,k_]=Sum[(a[n] t^n),{n,0,k}]
Cs=Collect[D[x[a,t,8],t,t]+4 x[a,t,8],t]
Sum[t^n a[n], {n, 0, k}]
4 a[0] + 2 a[2] + t (4 a[1] + 6 a[3]) +
t^2 (4 a[2] + 12 a[4]) + t^3 (4 a[3] + 20 a[5]) +
t^4 (4 a[4] + 30 a[6]) + 4 t^7 a[7] + t^5 (4 a[5] + 42 a[7]) +
4 t^8 a[8] + t^6 (4 a[6] + 56 a[8])
```

Now, we set the coefficients of t^n equal to zero, say for $n=1,2,\dots,6$. For clarity, we put them in the form of a table as follows:

```
TableForm[Table[{p,Coefficient[Cs,t,p]==0},{p,0,6}]]
0  4 a[0] + 2 a[2] == 0
1  4 a[1] + 6 a[3] == 0
2  4 a[2] + 12 a[4] == 0
3  4 a[3] + 20 a[5] == 0
4  4 a[4] + 30 a[6] == 0
5  4 a[5] + 42 a[7] == 0
6  4 a[6] + 56 a[8] == 0
```

Next we need to conjecture a recurrence relation for $a[n]$ and we are able to compute $a[0]$ and $a[1]$. In fact, since $x(0)=1$ and $x'(0)=0$, then $a[0]=1$ and $a[1]=0$. Furthermore, the recurrence relation is:

$$a[n]= -4 a[n-2]/(n(n-1)).$$

That conjecture is entered below to define the coefficient function $ai[j]$:

```
Clear[ai]
```

```

ai[0]=1
ai[1]=0
ai[j_]:= -4 ai[j-2]/(j(j-1))

1

0

```

Now we are in a position to find all the coefficients:

```
Table[ai[j],{j,20}]
```

```

{0, -2, 0,  $\frac{2}{3}$ , 0,  $-\left(\frac{4}{45}\right)$ , 0,  $\frac{2}{315}$ , 0,  $-\left(\frac{4}{14175}\right)$ , 0,  $\frac{4}{467775}$ , 0,
 $-\left(\frac{8}{42567525}\right)$ , 0,  $\frac{2}{638512875}$ , 0,  $-\left(\frac{4}{97692469875}\right)$ , 0,
 $\frac{4}{9280784638125}$ }

```

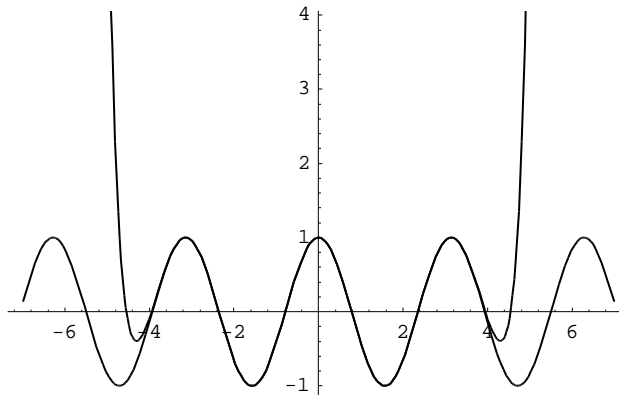
```
xi[t_,k_]:=x[ai,t,k]
xi[t,20]
```

$$\begin{aligned}
& 1 - 2t^2 + \frac{2t^4}{3} - \frac{4t^6}{45} + \frac{2t^8}{315} - \frac{4t^{10}}{14175} + \frac{4t^{12}}{467775} - \frac{8t^{14}}{42567525} + \\
& \frac{2t^{16}}{638512875} - \frac{4t^{18}}{97692469875} + \frac{4t^{20}}{9280784638125}
\end{aligned}$$

Observe now that if we use Mathematica to solve the differential equation, then the solution is $\cos(2t)$ for which the first twenty terms of its Maclaurin Series is exactly what we have obtained above.

To end let us compare the plot of the exact solution and the partial sum $\{t,20\}$ on the interval from -7 to 7 :

```
compare=Plot[{xi[t,20],Cos[2 t]},{t,-7,7}]
```



-Graphics-

The students observed that we have a very good fit from -4 to 4 but the approximation is very poor outside that interval. We would get a better approximation over a larger interval by taking more terms of the power series of $\text{Cos}(2t)$. However no matter how many terms are used, the approximation will be very poor outside some finite interval $[-s,s]$ because the exact solution is bounded between -1 and 1 for all t , whereas the partial sum $x[t,k]$ is a polynomial which must be unbounded.

```
TableForm[ai[t],{t.11,20}]
```

```
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```
General::stop:
  Further output of $RecursionLimit::reclim
  will be suppressed during this calculation.
```

```
ai[11]
```