Example 8.

Using Laplace Transforms, solve the following initial value problem:

\[ x''(t) + 4x(t) = \sin(3t); \ x(0) = 0 \text{ and } x'(0) = 0. \]

As a first step, we need to take the Laplace Transform of both sides of the equation. Unfortunately, for the left hand side, this cannot be done using Mathematica because some properties of the Laplace Transforms have to be applied. Using these properties we obtain for the left hand side:

\[ s^2 X(s) + 4 X(s) \]

where \( X(s) \) denotes the Laplace Transform of the unknown function \( x(t) \).

As for the right hand side Mathematica can be applied:

\[
\text{<<Calculus`LaplaceTransform`}
\text{LaplaceTransform}[\sin(3 t), t, s]
\]

\[
\frac{3}{9 + s^2}
\]

Thus, \( X(s) = \frac{(As+B)}{(s^2+4)} + \frac{(Cs+D)}{(s^2+9)} \). Equivalently we solve the system:

\[
\text{Solve}\left\{\begin{array}{l}
A + B = 0, \\
B + D = 0, \\
9A + 4C = 0, \\
9B + 4D = 3,
\end{array}\right\}, \{A, B, C, D\}
\]

\[
\left\{\begin{array}{l}
A \rightarrow -\frac{3}{5}, \\
B \rightarrow \frac{3}{5}, \\
C \rightarrow -\frac{27}{20}, \\
D \rightarrow -\frac{3}{5},
\end{array}\right\}
\]

Now we are in a position to find the solution to the initial value problem:

\[
\text{InverseLaplaceTransform}[\frac{3}{10} \frac{2}{s^2+4} - \frac{1}{5} \frac{3}{s^2+9}, s, t]
\]

\[
\frac{3 \sin(2t)}{10} - \frac{\sin(3t)}{5}
\]