COMPUTER TECHNOLOGY AND PROBLEM SOLVING

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1. Introduction. The advancement of technology during the past 10 years has greatly affected how we teach, what we teach and how our students learn in our undergraduate courses. Computer technology is greatly influencing the way we do mathematics and the problems which are accessable to our students. Before the introduction of Mathematica, Maple and other computer algebra systems, algebraic manipulation discouraged many students from studying mathematics and it is this reason that makes students think that mathematics is the most difficult subject. This dillusion keeps students from appreciating mathematics as a subject full of power and ideas and that it requires a lot of critical and logical thinking. By using computer algebra systems like Mathematica, Maple and Derive we can spend less time on doing algebraic manipulation, routine problem solving and sketching useless graphs, and spend more time on understanding concepts, and applying them to solve more realistic and challenging problems. The numerical and graphical capabilities of these software make students see what they could never before even imagine. These powerful capabilities can also help students make discoveries with our careful guidance. Many things which could not be done before now can be done easily with the help of computers. Many numerical methods which were very difficult to implement on a computer can now be done in meaningful and simple way with the help of computer algebra systems. With training in the use of computer technology in their mathematics classes, students are more capable of applying mathematics in their major fields and in the business world. Certainly the classroom experience will be more relevant to the "real world". In this paper we will use two types of problems that we can give to our calculus students to illustrate the computer can be used to aid in the solution of mathematics problems, how they will enhance students' understanding of calculus and improve their problem solving skills, why computer technology alone can not solve the problems and why mathematics reasoning is still of vital importance.

2. Tangent line problem. A typical problem given to students in the first semester of calculus is "Find an equation of the line tangent to the graph of $f(x) = x^2$ at the point $(2, f(2))$". With the advent pf computer algebra systems and their excellent graphics capabilities, a new type of tangent line problem is accessable to students taking calculus. We will begin by stating the problem in general, and then work out a specific example. The problem can be stated as follows: "Let $f(x)$ be a differentiable function. Under what conditions does there exist a differentiable function $g(x)$ such that the lines tangent to the graph of $y = g(x)$ are of the form $y = ax + f(a)$ for all a in the domain of f? If $g(x)$ exists, then find an expression for it in terms of $f(x)$". A complete discussion of this problem can be found in [1] and [2].

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Example: Let $f(x) = x^2$. Is there a differentiable function $g(x)$ whose tangent lines are of the form $y = ax + a^2$ for all $a \in R$? If so, what is $g(x)$?

Solution: We begin by using a computer algebra system to sketch lines of the form $y = ax + a^2$. The graphs of these lines can be generated by the programs listed below.

Maple program

```
> f := x-> x^2;
> f1 := a-> a*x + f(a);
> line := proc(low, high, step)
> c1 := { };
> for i from low by step to high do
> c1 := 'union'(c1, f1(i));
> od;
> end;
> line(-10, 10, 1/10) :
> plot(c1, x = -10..10, y = -10..10, scaling = CONSTRAINED);
```

Derive commands

```
vector(ax + a^2, a, -10, 10, 1/10)
Simplify
Plot
Plot
```

Mathematica program

```
Clear[t]
t = Table[a*x + a^2, a, -10, 10, 0.1];
Plot[Evaluate[t], {x, -10, 10}, PlotRange-> {-10, 10}, AspectRatio-> 1]
```

The picture shown on the computer suggests that the function $g(x)$ exists and $g(x)$ is a quadratic function of the form $g(x) = -\alpha x^2$, where $\alpha > 0$. The equation of the line tangent to the graph of $y = g(x)$ at the point $(\beta, g(\beta))$ is given by $y = -2\alpha\beta x + \alpha\beta^2$. This tangent line is of the form $y = ax + a^2$ if only if $a = -2\alpha\beta$ and $a^2 = \alpha\beta^2$. Since $\alpha > 0$ we see that these equations are satisfied when $\alpha = \frac{1}{4}$. Hence the function $g(x) = -\frac{1}{4}x^2$ has tangent lines of the form $y = ax + a^2$. We would like to point out that without the ability to visualize the set of lines of the form $y = ax + a^2$, this problem would be extremely difficult for students to solve. Hence the computer aids in the solution of this problem by allowing us to visualize the problem and then make a conjecture about the existence and form of the function $g(x)$ that we are trying to find.

3. Area Problem.

In our calculus course we have adopted the reform oriented Harvard Calculus book. Our intention is to teach the students not just the rules and procedures of differentiation and integration, but more about the concepts, geometric and numerical aspects, and real world applications of calculus. With the development of advanced computer algebra systems like Mathematica, students are able to reinforce their understanding of calculus concepts and ideas by solving more challenging, realistic and meaningful problems. We want to train our students to be problem solvers. One type of problem every student who takes calculus should be able to solve is finding area of a region in a plane bounded by curves. In Mathematica finding the area of the region bounded by the curve with the equation $x^4 + y^4 = a^4$ is not much more difficul than finding the area of the region bounded by the circle with the equation $x^2+y^2 = a^2$. Both can be done by evaluating a definite integral. What about finding the area of the region bounded by the curve whose equation is $x^{100} + y^{100} = a^{100}$? Is it a difficult task? This depends on if you use computer technology. Without using a computer algebra system or computer programming or a graphing calculator this would be an insurmountable task. With Mathematica students can find the area easily by numerically evaluating the integral $\int_{-a}^{a}(a^{100}-x^{100})^{\frac{1}{100}}\,dx$ for a specific value of a and doubling it. With proper training it is now possible for our students to make discoveries. From the graph drawn by Mathematica they observe that the graph of $x^{100} + y^{100} = a^{100}$ is almost a square. Hence the area inside the curve $x^{100} + y^{100} = a^{100}$ is almost $4a^2$.

Now let us show our students that Mathematica can be used to solve more complicated and difficult area problems. One of these problems is to find the area of the region that consists of the points whose total distance from the vertices of a unit square is less than or equal to $2 + \sqrt{2}$. It is our belief that this problem cannot be done by pencil and paper method. It is not easy even with assistance of Mathematica. The reason is that the y-coordinate cannot be explicitly in terms of the x-coordinate. For this problem just understanding the concept of integral and the connection between area and integral is not near enough, critical thinking and problem solving skills are necessary. How can we solve the problem? The first step is to get the picture of the region. Let us place the square in the rectangular coordinate plane so that the vertices of the square are the points $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. Then one can see, using distance formula, that the region is bounded by the curve whose equation is

$$\sqrt{x^2+y^2} + \sqrt{x^2+(y-1)^2} + \sqrt{(x-1)^2+y^2} + \sqrt{(x-1)^2+(y-1)^2} = 2+\sqrt{2}.$$

One may use CountourPlot to sketch the curve. In order to find the area, one proceeds to find values of y for some values of x and use a Riemann sum or the trapezoidal rule to get an approximate value of the area. A different approach may be used to sketch the curve and find the area. In this approach we introduce parameter and derive parametric equations that correspond to the rectangular equation. Both x-coordinate and y-coordinate are functions of a parameter. Hence one can express the area of the region as an explicit definite integral can be computed by built-in numerical method. The program written in Mathematica used to sketch the curve and compute the area is given below:

$Clear[x, y, s, t, a, b, c, e, f, g, g1, g2, g3, g4, g5, g6, g7, g8, area]$
$s = 2 + Sqrt[2] - t;$
$a = 4 * (t^2 - s^2);$
$b = -8 * s^2 * (t^2 - 1);$
$c = 4 * s^2 * (t^2 - 1) + (t^2 - 1)(s^2 - 1)(t^2 - s^2);$

$x[t\_] := Evaluate[N[(-b - Sqrt[b^2 - 4*a*c])/(2*a)]];$
$e = 4(t^2 - 1);$
$f = -4(t^2 - 1);$
$g = 4*t^2*(x[t])^2 - (t^2 - 1)^2;$
$y[t\_] := Evaluate[N[(-f + Sqrt[f^2 - 4*e*g])/(2*e)]];$
$g1 = ParametricPlot[\{x[t], y[t]\}, \{t, 1.00001,$
$(2 + Sqrt[2])/2 - 0.00001\}, AspectRatio -> 1];$
$g2 = ParametricPlot[\{x[t] + 0.5, y[(4 + Sqrt[2])/2 - t]\},$
$\{t, 1.00001, (2 + Sqrt[2])/2 - 0.00001\}];$
$g3 = ParametricPlot[\{y[t], x[t]\},$
$\{t, 1.00001, (2 + Sqrt[2])/2 - 0.00001\}];$
$g4 = ParametricPlot[\{y[(4 + Sqrt[2])/2 - t], x[t] + 0.5\},$
$\{t, 1.00001, (2 + Sqrt[2])/2 - 0.00001\}];$
$g5 = ParametricPlot[\{x[t], 1 - y[t]\},$
$\{t, 1.00001, (2 + Sqrt[2])/2 - 0.00001\}];$
$g6 = ParametricPlot[\{x[t] + 0.5, 1 - y[(4 + Sqrt[2])/2 - t]\},$
$\{t, 1.00001, (2 + Sqrt[2])/2 - 0.00001\}];$
$g7 = ParametricPlot[\{1 - y[t], x[t]\},$
$\{t, 1.00001, (2 + Sqrt[2])/2 - 0.00001\}];$
$g8 = ParametricPlot[\{1 - y[(4 + Sqrt[2])/2 - t], x[t] + 0.5\},$
$\{t, 1.00001, (2 + Sqrt[2])/2 - 0.00001\}];$
$Show[g1, g2, g3, g4, g5, g6, g7, g8]$
$area = 8*NIntegrate[(y[t] - 1)*x'[t], \{t, 1, (2 + Sqrt[2])/2\}] + 1$

References

1. S. Ligh and R. Wills, *On Linear Functions II*, Mathematics and Computer Education **29, No. 1** (1995), 32–52.

2. S. Ligh and R. Wills, *On Linear Functions III*, Mathematics and Computer Education **30, No. 1** (1996), 9–18.