CHAOS IN THE CLASSROOM

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One of the products of the calculus reform movement has been the "Rule of Three." This is the idea that mathematical concepts should be examined from a graphical, numerical, and a symbolic point of view. At the same time there has been an emphasis on discovery techniques and the incorporation of technology. To illustrate this we developed a series of activities that were used in a workshop for high school and middle school mathematics teachers. The activities, which were based on the treatment of Chaos in [1], turned out to be an excellent example of this approach.

§1 Dynamical Systems

Let $F(x)$ be a function. An equation of the form

$$x_{n+1} = F(x_n)$$

is a *discrete dynamical system*.

The main objective of the series of activities was to investigate the family of dynamical systems represented by the function $F(x) = x^2 + c$. We did this by examining the behavior of the sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$. (called the *orbit*) for various values of *c* and for various starting values x_0 (called the *seed* of the orbit).

In our early investigations the TI calculators were extremely useful. (We had both TI-81 and TI-82 calculators.) The orbit of a dynamical system can easily be computed by implementing the following sequence of steps:

(1) After pressing the "y=" button on the calculator enter the function $F(x)$ in as Y_1 .
(2) After quitting from the "y=" menu type in the seed and press "enter".
(3) Type in Y_1(ANS) and press "enter". (Y_1 is in the "Y-VARS" menu).
(4) Continue to press enter to generate as much of the orbit as you like.

Using these techniques we were able to discover examples of *fixed orbits* (orbits for which $F(x_0) = x_0$), *periodic orbits* (orbits for which , for some n, $F^n(x_0) = x_0$.), *eventually fixed orbits* (for some k, $F^k(x_0)$ is a seed for a fixed orbit), and *eventually periodic orbits* (for some k, $F^k(x_0)$ is a seed for a periodic orbit). We were even able to explore some examples of chaotic behavior without defining it . (In fact we never did define it.)

§2 Graphical Analysis

After examining dynamical systems from the numerical point of view, we turned to the graphical. One way of analyzing the behavior of a discrete dynamical system is through *graphical analysis.* To see how this works in practice follow the instructions below to analyze the orbit of the dynamical system corresponding to $F(x) = x^3$.

1. On a coordinate system sketch a careful graph of $y = x^3$ along with a graph of the line $y = x$.
2. We will begin with a seed of x_0=0.9. To find the orbit of x_0 we begin at the point (x_0, x_0) on the line $y = x$. Draw a vertical line segment from there to the point (x_0, x_1). You should now be on the curve $y = x^3$.
3. Continue by drawing a horizontal line segment back to the line $y = x$ and you will be at the point (x_1, x_1). Follow this with another vertical line segment to $y = x^3$, then another horizontal to $y = x$. You should now be at (x_2, x_2). Continue in this way to locate (x_k, x_k) for $k = 3,4,\ldots$ until you can describe what is happening to the orbit.
4. Use graphical analysis to characterize the orbit of every possible seed.

By applying graphical analysis to a number of examples we were led to the conclusion that the *fixed points*, intersections of the graph of $y = F(x)$ with the line $y = x$, were crucial to the understanding of the dynamical system. we also noticed that there were two important categories of fixed points, *attracting fixed points* (those that attract nearby orbits) and *repelling fixed points* (those that repel nearby orbits).

One particularly illuminating investigation was the examination of the dynamical systems corresponding to linear functions. Through graphical analysis the workshop participants were able to completely characterize the behavior of linear systems. (When the absolute value of the slope is less than (respectively, greater than) one the fixed point is attracting (respectively, repelling).)

§3 A Quadratic Family

At this point we began to concentrate our attention on the family of dynamical systems that correspond to the function $F(x) = x^2 + c$ for c a real number. The first step was to investigate the fixed points of this system. This was the symbolic stage of the investigation as the participants were asked to derive the general formula for these fixed points. By using the quadratic formula they were able to determine that the fixed points were:

$$p_+ = \frac{1}{2}\left(1 + \sqrt{1-4c}\right)$$
and
$$p_- = \frac{1}{2}\left(1 - \sqrt{1-4c}\right)$$

In particular there are no fixed points when $c > \frac{1}{4}$ and exactly one fixed point when $c = \frac{1}{4}$. When $c < \frac{1}{4}$ there are exactly two fixed points. (This phenomenon, in which the system changes from one fixed point to two fixed points at $c = \frac{1}{4}$ is described by saying that there is a *bifurcation* at $c = \frac{1}{4}$.) Previous experimental evidence had already indicated something of the sort, but the precise determination of the fixed points confirmed and refined the observations.

The next step was to determine when there was another bifurcation in which the one attracting fixed point, p_-, bifurcated into a cycle of period 2. This is gotten by solving the equation $F^2(x_0) = x_0$. The solution of this equation involves factoring a fourth degree polynomial, using the fact that two of the roots, p_+ and p_-, are already know. This gave them a good workout in concepts from high school algebra that were a crucial step in analyzing the system. The two points in the period two orbit turn out to be:

$$q_+ = \frac{1}{2}\left(-1 + \sqrt{-4c-3}\right)$$

and

$$q_- = \frac{1}{2}\left(-1 - \sqrt{-4c-3}\right)$$

From this it is clear that the period two orbits only occur when $c < -\frac{3}{4}$ so that the next bifurcation occurs at $c = -\frac{3}{4}$.

§4 Computer Experiments

The final stages of the activity occurred on the computer. They were provided with a TrueBasic program that would iterate the dynamical system and even to illustrate the corresponding graphical analysis. (After writing one myself, I discovered that a similar one was provided in the appendix of [1].) They used this program to begin to construct their own *orbit diagram*, for the quadratic family $F(x) = x^2 + c$. The orbit diagram plots, for each value of c, the points (c, x) where x is a point in an eventual orbit resulting from the seed $x_0 = 0$.

As the culmination of their endeavors the workshop participants were given a program to run to produce the orbit diagram and compare it to the one that they had predicted. It was again written in TrueBasic and was a slight modification of another program appearing in the appendix of [1]. The output appears in figure 1. From this diagram they were able to observation the predicted bifurcations at $c = \frac{1}{4}$ and $c = -\frac{3}{4}$ and observe the many other bifurcations that occur in the transition to chaotic behavior.

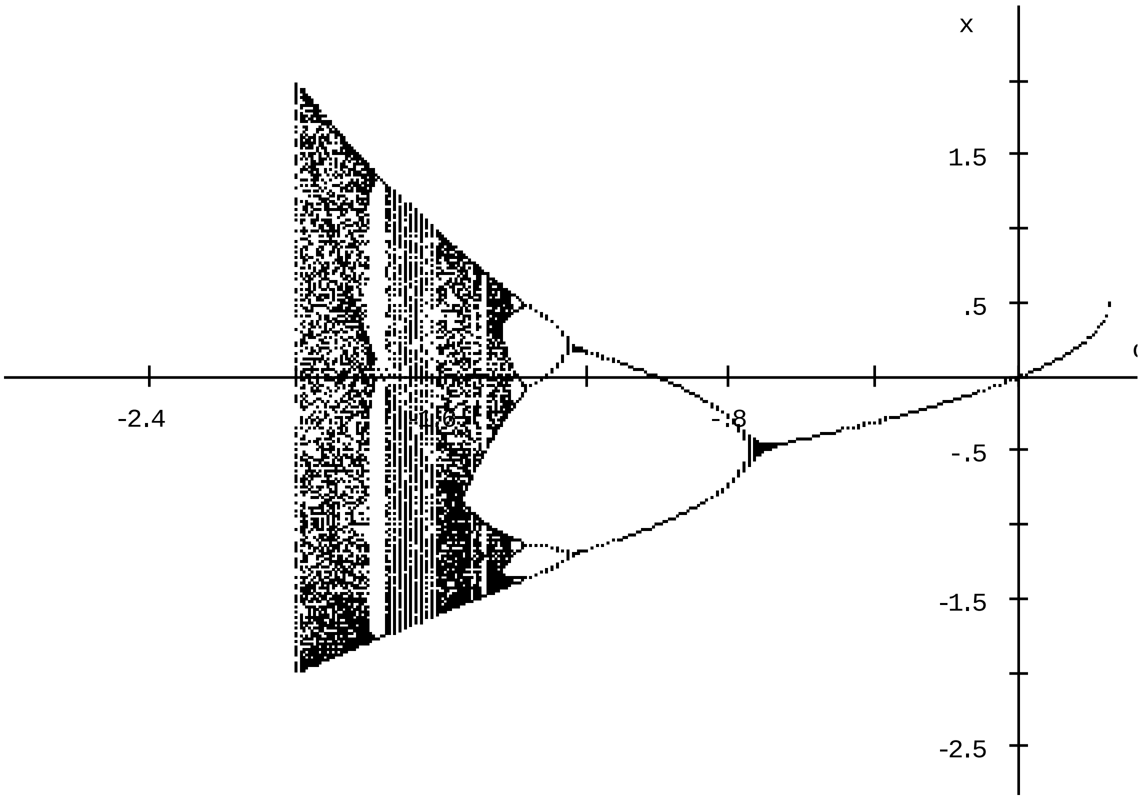

Figure 1 Orbit diagram for $F(x) = x^2 + c$

§5 Conclusion

We found the investigation of discrete dynamical systems to be a marvelous topic for high school level mathematics. It was engaging and used a lot of important mathematical techniques. In addition it is ideally suited for the discovery approach to learning (especially as presented by Devaney in [1]) and the incorporation of technology.

Reference

[1] Devaney, Robert L., *A First Course in Chaotic Dynamical Systems*, Addison-Wesley, 1992