TRIGONOMETRIC REPRESENTATION OF $[x]$

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1. Introduction

There have been much discussions in recent years about the teaching of calculus. Among the various directions in the reform movement is the Consortium based at Harvard University. The text [2] for this project begins with a discussion of a library of functions. We introduced in our first semester calculus class several functions that are applicable to students' environment. The postage stamp and the grading functions are examples of step functions, such as the greatest integer function. This paper gives an account of our experience about what we have benefited from using the Computer Algebra System, *Derive*, in the study of the greatest integer function. In fact, the use of *Derive* led us to mathematical inquiry and discovery instead of just as a tool to reduce tedious calculation or to provide visualization.

In order to study the greatest integer function in *Derive (version 2.0)*, one must first load the utility file MISC.MTH. This file contains the function FLOOR(a, b) which is defined as the greatest integer less than or equal to a/b. See *Derive* user manual [4, p.198]. Therefore, the greatest integer function, denoted by $[x]$, is given as FLOOR(x,1). When we use the *Derive* command "Simplify" on this function, we get the following representation for $[x]$:

$$[x] = \frac{1}{\pi}\tan^{-1}(\cot\pi x) + x - \frac{1}{2} \qquad (1)$$

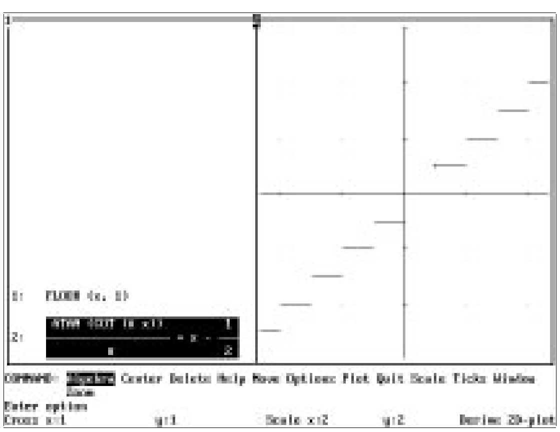

Figure 1. Graph of $y = \frac{1}{\pi}\tan^{-1}(\cot\pi x) + x - \frac{1}{2}$

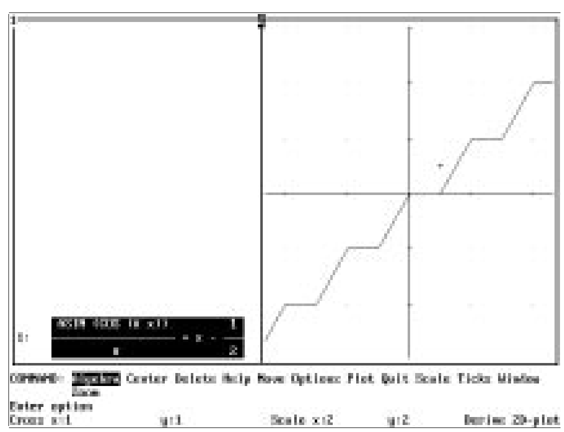

Figure 2. Graph of $y = \frac{1}{\pi}\sin^{-1}(\cos\pi x) + x - \frac{1}{2}$

as shown in Figure 1. The above equality can be proved by means of properties of trigonometric functions for non-integral x. Figure 1 also gives the graph of (1). One can see that it is a "step" function which assumes only one value on each interval.

2. Trigonometric Identities

Equality (1) motivated us to study functions of the type:

$$y = \frac{1}{\pi} f^{-1}(f_{\mathrm{c}}(\pi x)) + x - \frac{1}{2} \tag{2}$$

where f is a trigonometric function with f^{-1} and f_c being its inverse and cofunction respectively. As an illustration, let us consider the function:

$$y = \frac{1}{\pi} \sin^{-1}(\cos \pi x) + x - \frac{1}{2}. \tag{3}$$

One may conjecture that (3) is also a step function. A *Derive* plot shows that this is not the case. However, we obtain a very interesting graph which consists of horizontal line segments and slanted line segments with slope 2 (see Figure 2). For any integer n, (3) can be rewritten as follows:

$$\frac{1}{\pi} \sin^{-1}(\cos \pi x) + x - \frac{1}{2} = \begin{cases} [x] & \text{if } 2n \leq x < 2n+1; \\ 2x - ([x]+1) & \text{if } 2n+1 \leq x < 2n+2. \end{cases}$$

This representation of (3) accounts for the shape of the graph, and its derivation is given in [1]. The graphs of (2) with other four trigonometric functions are given in Figure 3 to 6, and they are translations of the graphs of (1) or (3).

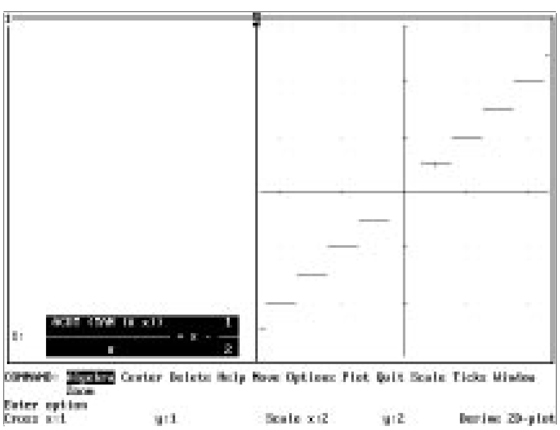

Figure 3. Graph of $y = \frac{1}{\pi}\cot^{-1}(\tan \pi x) + x - \frac{1}{2}$

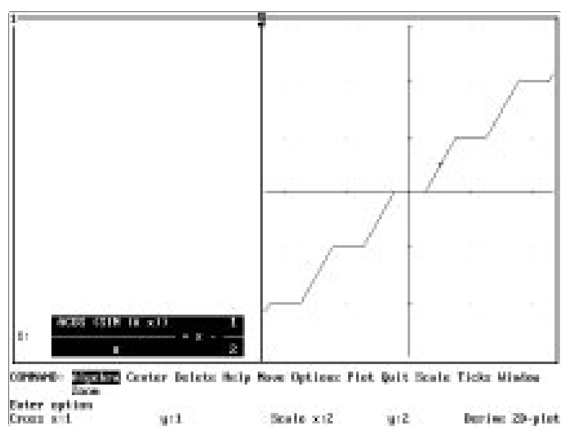

Figure 4. Graph of $y = \frac{1}{\pi}\cos^{-1}(\sin \pi x) + x - \frac{1}{2}$

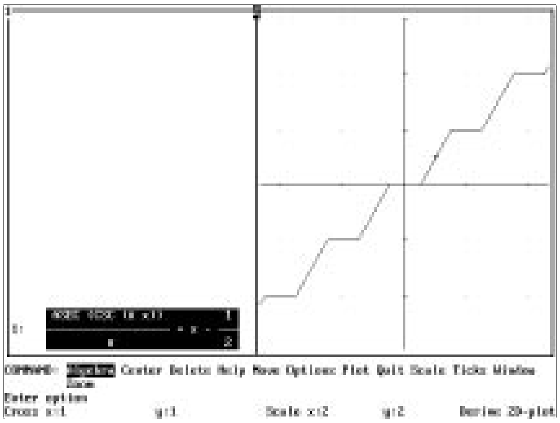

Figure 5. Graph of $y = \frac{1}{\pi}\sec^{-1}(\csc\pi x) + x - \frac{1}{2}$

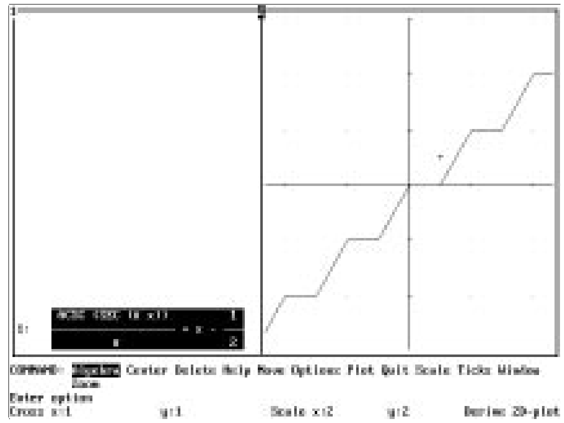

Figure 6. Graph of $y = \frac{1}{\pi}\csc^{-1}(\sec\pi x) + x - \frac{1}{2}$

Here, we would like to list five other trigonometric representations for $[x]$. The proofs of these equalities are also given in [1].

$$[x] = \frac{(-1)^{[x]}}{\pi}\sin^{-1}(\cos\pi x) + x - \frac{1}{2}.$$

$$[x] = (-1)^{[x]+1}\left(\frac{1}{\pi}\cos^{-1}(\cos\pi x) - \frac{1}{2}\right) + x - \frac{1}{2}$$

$$[x] = -\frac{1}{\pi}\cot^{-1}(\cot\pi x) + x,\ x \neq n$$

$$[x] = \frac{(-1)^{[x]}}{\pi}\csc^{-1}(\sec\pi x) + x - \frac{1}{2},\ x \neq n + \frac{1}{2}$$

$$[x] = (-1)^{[x]+1}\left(\frac{1}{\pi}\sec^{-1}(\sec\pi x) - \frac{1}{2}\right) + x - \frac{1}{2},\ x \neq n + \frac{1}{2}$$

3. Integrals of $[x]$

In evaluating definite integrals of $[x]$ using *Derive (version 2.0)*, we "Author" the integral, for instance $\int_2^5 [x]\,dx$, and then "Simplify." The answer 16 is incorrect! For the indefinite integral $\int [x]dx$, *Derive* gives

$$\int [x]dx = x\left(\frac{1}{\pi}\tan^{-1}(\cot\pi x) + x - \frac{1}{2}\right) + C.$$

By combining the above equation with (1), we have $\int [x]dx = x[x] + C$. We note that this formula gives $\int_2^5 [x]\,dx = 21$ which is also incorrect. These discrepancies motivated us to investigate the integral $I = \int_a^b [x]\, dx$ for real numbers a and b. By using the equality (1) and the substitution $u = \frac{1}{\pi}\tan^{-1}(\cot \pi x)$, we acquired the following integration formula:

$$\int_a^b [x]\, dx = \frac{-\ [x]^2\ +\ 2x[x]\ -\ [x]}{2}\Big]_a^b.$$

The above formula was also obtained by Sy [3] using a different argument.

REFERENCES

1. Fung, D. and Ligh, S., *"Trigonometric Representations of the Greatest Integer Function from Derive,"* submitted for publication.

2. Hugues-Hallett, D., Gleason A., et al., *Calculus*, John Wiley & Son, 1994.

3. Sy, S. W., *"An Indefinite Integral for [x],"* Pi Mu Epsilon Journal, Volume 9, Number 10, Spring 1994, p. 683-684.

4. User Manual, *Derive Version 2*, Fourth edition, Soft warehouse, Honolulu, Hawaii, 1991.