Conceptual vrs. Procedural Knowledge in Introductory Calculus - Programming Effects

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Student Knowledge. Evidence abounds that many calculus students are learning to execute tasks in a routine, algorithmic fashion while lacking very basic conceptual understanding of the topics being studied. For example, Table I gives results of several final exam questions given to approximately 200 students who were completing an introductory calculus course. The examples in this table demonstrate that we are having remarkable success on algorithmic problems. Some would say that with a little more practice we might insure a short life span for the HP-28C.

<table>
<thead>
<tr>
<th>Question</th>
<th>Percent of Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Find $dy/dx$: $e^{x+y} = \sin y$</td>
<td>86%</td>
</tr>
<tr>
<td>2. Find $dy/dx$: $y = x^7 \cos(\pi x)$</td>
<td>92%</td>
</tr>
<tr>
<td>3. Integrate: $\int x^2 \sin x , dx$</td>
<td>89%</td>
</tr>
<tr>
<td>4. Integrate: $\int \frac{3}{x(x^2 + 1)} , dx$</td>
<td>73%</td>
</tr>
<tr>
<td>5. Find the first three terms of the</td>
<td>90%</td>
</tr>
<tr>
<td>Taylor series for $f(x) = \sqrt[4]{x}$ about 1.</td>
<td></td>
</tr>
<tr>
<td>6. Find the area of the region bounded by $y = 4x - x^2$ and $y = x^2$</td>
<td>86%</td>
</tr>
</tbody>
</table>

The same student subjects reported on in Table I were given a Calculus Concept Test during the final week of the semester. Three of the problems on this test are reported in Table II. The results on these problems suggest that many students do not recognize the derivative presented in symbolic form, and that they are unable to formulate or interpret geometric representations of fundamental concepts. Not only did students have conceptual difficulty with fundamental concepts of calculus, but problem 2(a) indicates that almost 40% of these students had conceptual difficulty at an even more basic level. These students were unable to correctly interpret the relationship between a function and its graph. This suggests that many of the intuitive explanations that instructors use in the classroom may be of little benefit for many students. The results on problem 3 indicate that when students had no
### TABLE II
Calculus Concept Test

<table>
<thead>
<tr>
<th>Question</th>
<th>Percent Correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Suppose ( f, f' ) and ( f'' ) are differentiable and that ( f(5) = 9 ), ( f'(5) = 4 ) and ( f''(5) = -1 ). Find ( \lim_\limits{h \to 0} \frac{f(5 + h) - f(5)}{h} ).</td>
<td>22%</td>
</tr>
<tr>
<td>2. Suppose line ( L ) is tangent to the curve ( y = f(x) ) at the point ( (5, 3) ) as indicated at the right. Find ( (a) ) ( f(5) ), ( (b) ) ( f'(5) ).</td>
<td>63%, 22%</td>
</tr>
<tr>
<td>3. Evaluate ( \int_{-1}^{2}</td>
<td>x</td>
</tr>
</tbody>
</table>

"formula", they had nothing to fall back on. Their concept of the Riemann integral did not allow them any alternative approach to the problem.

The examples that have been examined tend to confirm the contention that even when students are emersed in the study of calculus, our instructional emphasis results in conceptual knowledge that is indeed minimal. The evidence may also suggest that many students have difficulty in relating geometric explanations to the corresponding algebraic or symbolic representations of fundamental concepts.

A Programming Example. Because many students are now entering colleges and universities with experience in a programming language, we believe that having students write computer programs could be helpful in developing conceptual schemas related to the fundamental ideas of the introductory calculus course. We have instituted a one-credit supplemental course in computational calculus where students write their own programs to find limits, right and left hand derivatives, Riemann integrals, solutions to equations, etc. These programs, used in conjunction with graphics utilities (which students do not write) are used by the students to investigate several questions, among them the existence of derivatives and integrals for various functions at specified points or on specified intervals. Other types of questions are also investigated. It is quite easy, for example, to consider functions with discontinuities and examine the limit of Riemann sums when using left endpoints, right endpoints, midpoints or randomly selected points in the subintervals. By running a few examples, and examining the numerical output, students are easily convinced that the integral is independent of the points selected.
Table III lists several questions that were given on the Calculus Concept Test to those same students referred to earlier in this paper, and also to a group of 24 students who completed the computational calculus course. As can be seen from these results, students who were enrolled in the supplemental course appear to have developed richer schema related to both the derivative and integral concepts.

Justification of programming. Three seemingly logical arguments are often advanced for justifying the use of "canned programs" in calculus instruction. One argument draws a parallel between the discrete nature of computers and students' experiences with discrete situations. Discrete numerical output belongs to the students' world, and is more likely to be properly interpreted than are the symbolic representations of continuous functions. Second, the graphics capability, and in particular the ability to

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Concept Test (Percent Correct)} & \textbf{Regular Class} & \textbf{Regular + Programming Class} \\
\hline
\textbf{Question} & & \\
\hline
1. Suppose $f$, $f'$, and $f''$ are differentiable and that $f(5) = 9$, $f'(5) = 4$ and $f''(5) = -1$. Find $\lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$ & 22\% & 63\% \\
\hline
2. Suppose line $L$ is tangent to the curve $y = f(x)$ at the point (5, 3) as indicated at the right. Find $f(5)$ & 63\% & 88\% \\
\hline
(b) $f'(5)$ & 22\% & 46\% \\
\hline
3. Evaluate $\int_{-1}^{2} |x| \, dx$ & 7\% & 50\% \\
\hline
4. If $f(x) = (x + 1)^{10}$, find $\lim_{h \to 0} \frac{f(h) - f(0)}{h}$ & 8\% & 42\% \\
\hline
5. What is the maximum slope of the curve $f(x) = -x^3 + 3x^2 + 9x - 27$ & 6\% & 46\% \\
\hline
\end{tabular}
\end{table}
6. Suppose P is a partition of $[0, \pi/2]$ into $n$ subintervals, and $u_i$ is an arbitrary
point in the $i$th subinterval $[x_{i-1}, x_i]$. Explain why
$$\lim_{||P|| \to 0} \sum_{i=1}^{n} (\cos u_i) \Delta x_i = 1.$$