Napoleon-Like Properties of Spherical Triangles

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Abstract

If equilateral triangles are constructed outwards or inwards on the sides of any given triangle, then the centroids of these triangles are the vertices of an equilateral triangle. In elementary Euclidean geometry this result is known as Napoleon's Theorem.

Consider the following generalization of this construction. Let $d(\,\cdot\,,\,\cdot\,)$ denote Euclidean distance and suppose A, B, C are the vertices of any given, positively oriented triangle. Let point X be located s units from A along line AB and t units perpendicular to line AB. Assume s, t are directed distances with s measured positively from A to B and t positive when measured outward from triangle ABC. With the same sign conventions, the point Y is located $s \cdot d(B,C)/d(A,B)$ units from point B along line BC and $t \cdot d(B,C)/d(A,B)$ units perpendicular to line BC. Similarly, point Z is located $s \cdot d(\mathrm{C},A)/d(A,B)$ units from point C along line CA and $t \cdot d(C,A)/d(A,B)$ units perpendicular to line CA. In this way the points X, Y, Z are proportionately positioned relative to the points A, B, C. Note that X, Y, Z are the centroids mentioned in Napoleon's Theorem when $s = d(A,B)/2$ and $t = \pm\, d(A,B)/(2\sqrt{3})$. Are there other real numbers s, t for which triangle XYZ is equilateral? The answer to this question can be discovered by most any college geometry student aided with a computer algebra system (CAS).

In this paper we adapt the foregoing construction to certain classes of spherical triangles and use a CAS to determine various values of s, t with the properties given above.

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In Euclidean geometry Napoleon's theorem states that if equilateral triangles are constructed outwards or inwards on the sides of any given triangle, then the centroids of these triangles are the vertices of an equilateral triangle [**2**]. Consider the following generalization[1] of this construction. Suppose $\triangle ABC$ is any given, positively oriented triangle and let the point X be located s units from A along the line $\overleftrightarrow{AB}$ and t units perpendicular to $\overleftrightarrow{AB}$. Assume s, t are directed distances with s measured positively from A to B and t positive when measured outward from $\triangle ABC$. If a, b, c are the lengths of the sides opposite vertices A, B, C, respectively, and if the same sign convention is used, then the point Y is located $s_2 := s \cdot (a/c)$ units from B along $\overleftrightarrow{BC}$ and $t_2 := t \cdot (a/c)$ units perpendicular to $\overleftrightarrow{BC}$. See Figure 1. Similarly, point Z is located $s_3 := s \cdot (b/c)$ units from C along $\overleftrightarrow{CA}$ and $t_3 := t \cdot (b/c)$ units perpendicular to $\overleftrightarrow{CA}$. In this way the points X, Y, Z are proportionately positioned relative to the points A, B, C.

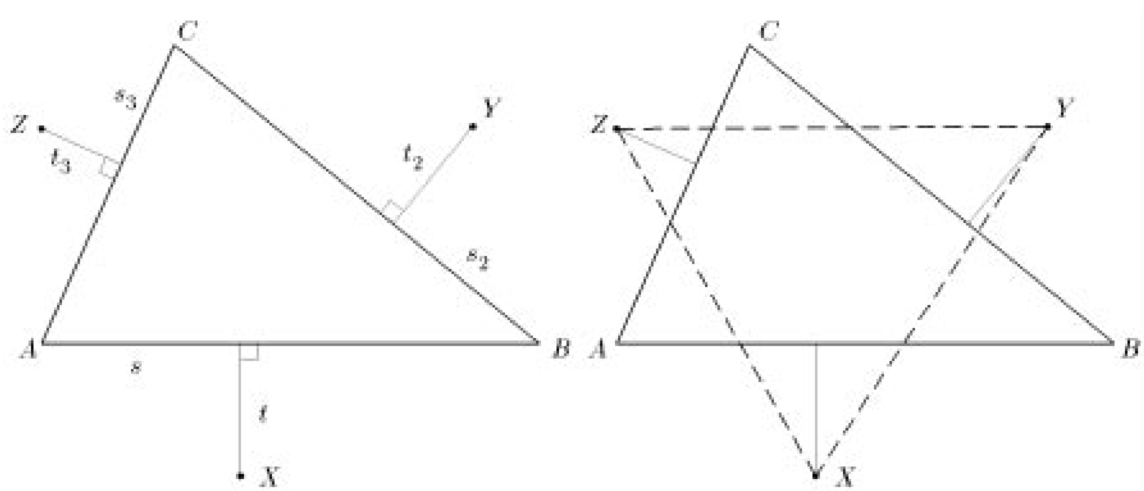

Figure 1. Construction of $\triangle XYZ$.

The points X, Y, Z so constructed yield the centroids mentioned in Napoleon's theorem when $s = c/2$ and $t = \pm c/\left(2\sqrt{3}\right)$. A natural question is: Are there any other values of s, t for which $\triangle XYZ$ is equilateral? Remarkably, if $\triangle ABC$ is not equilateral, then no other real numbers s, t make $\triangle XYZ$ equilateral. If $\triangle ABC$ is equilateral, then $\triangle XYZ$ is equilateral for

[1] I thank George Kung for suggesting this construction.

all real numbers s, t. These results are not hard to demonstrate by using a computer algebra system (CAS).

To illustrate, note that properties of similar triangles imply that, without loss of generality, we can assume $c = 1$. Let α be the measure of $\angle A$ and coordinatize A, B, C as $A = (0, 0)$, $B = (1, 0)$ and $C = (b\cos\alpha, b\sin\alpha)$ where $b > 0$ and $0 < \alpha < \pi$. Then the point X is $X(s, t) = (s, -t)$. The point Y can be obtained by a sum of three vectors, illustrated by arrows in Figure 2.

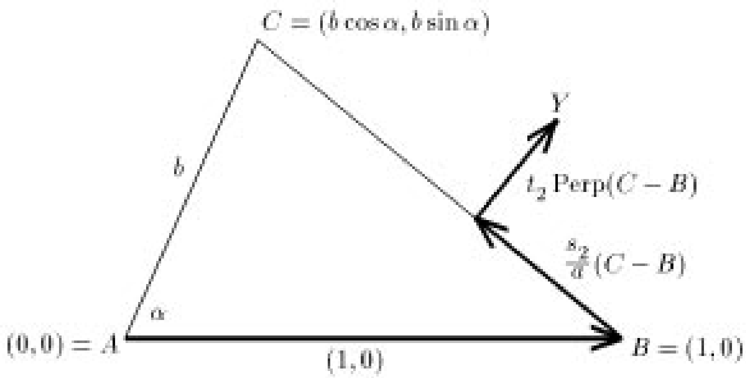

Figure 2. The point Y as a vector sum.

The unit vector $\mathrm{Perp}(C - B) := \frac{1}{a}(b\sin\alpha, 1 - b\cos\alpha)$ is an outer normal to $\triangle ABC$ at side $\overline{BC}$. Consequently we can write

$$\begin{aligned} Y(b, \alpha, s, t) &= B + \tfrac{s_2}{a}(C - B) + t_2\,\mathrm{Perp}(C - B) \\ &= (1 - s + bs\cos\alpha + bt\sin\alpha, t - bt\cos\alpha + bs\sin\alpha) \end{aligned}$$

since $s_2 = as$ and $t_2 = at$. Similarly, $\mathrm{Perp}(A - C) := (-\sin\alpha, \cos\alpha)$ is an outer unit normal to $\triangle ABC$ at side $\overline{AC}$, $s_3 = bs$, and $t_3 = bt$ whereby point Z is the vector sum

$$\begin{aligned} Z(b, \alpha, s, t) &= C + \tfrac{s_3}{b}(A - C) + t_3\,\mathrm{Perp}(A - C) \\ &= \Big(b(1 - s)\cos\alpha - bt\sin\alpha, bt\cos\alpha + b(1 - s)\sin\alpha\Big). \end{aligned}$$

Now $\triangle XYZ$ is equilateral if and only if:

$$\begin{cases} F(b, \alpha, s, t) := \|X - Y\|^2 - \|X - Z\|^2 = 0 \\ G(b, \alpha, s, t) := \|X - Y\|^2 - \|Y - Z\|^2 = 0 \end{cases} \tag{1}$$

Solutions of this nonlinear system of equations can be found by using a CAS. *Mathematica* [**3**], for example, offers $s = 1/2$ and $t = \pm 1/\left(2\sqrt{3}\right)$ as solutions[2] for all values of b and α. Inspection of the system (1) reveals that these are the only values of s, t that work for all choices of b and α, i.e. for all triangles ABC with $c = 1$. It follows from similar triangles that only $s = c/2$ and $t = \pm c/\left(2\sqrt{3}\right)$ make $\triangle XYZ$ equilateral for any

[2] *Mathematica* code for this and subsequent calculations can be found in the notebook file published electronically with this paper.

triangle ABC and for these values of s, t the points X, Y, Z are the centroids of equilateral triangles constructed outwards or inwards on the sides of $\triangle ABC$. This agrees with the conclusion of Napoleon's theorem.

If $\triangle ABC$ is equilateral, then $a = b = c = 1$ and $\alpha = \pi/3$ in the analysis above and *Mathematica* gives $F(1, \pi/3, s, t) = 0 = G(1, \pi/3, s, t)$ for all real numbers s, t. In this case, therefore, $\triangle XYZ$ is always equilateral.

The Spherical Case

What if $\triangle ABC$ is a spherical triangle? Although a spherical triangle has a centroid where the medians intersect, in general Napoleon's theorem does not hold on the sphere. However, do values of s, t still exist that make $\triangle XYZ$ equilateral? How can a CAS be used to investigate this question? We take a somewhat naïve approach and present some preliminary calculations and results.

Assume $\triangle ABC$ is a spherical triangle on the unit sphere $\mathbb{S}^2$ so that its vertices A, B, C are linearly independent unit vectors in $\mathbb{R}^3$. Let $\triangle ABC$ be positively oriented, i.e. the matrix with first row A, second row B, and third row C has positive determinant. Suppose $\times$ denotes vector cross product on $\mathbb{R}^3$ and that $\langle \cdot\,, \cdot \rangle$ is the Euclidean inner product with induced norm $\| \cdot \|$. By analogy with the plane case we make the following definitions:

$$A := (1, 0, 0) \quad B := (\cos c, \sin c, 0) \quad C := (\cos b, \sin b\ \cos\alpha, \sin b\ \sin\alpha)$$

The vertices of $\triangle ABC$. Assume $0 < a < \pi,\ 0 < c < \pi,\ 0 < b < \pi$, and $0 < \alpha < \pi$.

$$X(c, b, \alpha, s, t) := (\cos t \cos s, \cos t \sin s, -\sin t)$$

This point has latitude $-t$ and longitude s.

$$\text{Pole}(B, C) := (C \times B) \Big/ \|C \times B\|$$

Gives a pole of the great circle that contains B and C.

$$\text{Perp}(B, C) := (C - \langle B, C\rangle B) \Big/ \|C - \langle B, C\rangle B\|$$

On the great circle containing both B and C, this gives the point whose directed distance from B towards C is $\pi/2$.

$$\text{FarPt}(B, C, s_2) := (\cos s_2)B + (\sin s_2)\text{Perp}(B, C)$$

On the great circle containing both B and C, this gives the point whose directed distance from B towards C is s_2.

$$Y(c, b, \alpha, s, t) := \text{FarPt}\left(\text{FarPt}(B, C, s_2), \text{Pole}(B, C), t_2\right)$$

See Figure 3. Since there are no similar triangles on the sphere we cannot assume $c = 1$. Note that $s_2 = \frac{a}{c}\, s$ and $t_2 = \frac{a}{c}\, t$ where $a = \arccos\langle B, C\rangle$ and $c = \arccos\langle A, B\rangle$.

$Z(c, b, \alpha, s, t) \coloneqq \text{FarPt}\left(\text{FarPt}(C, A, s_3), \text{Pole}(C, A), t_3\right)$

The construction is similar to that for Y. Here $s_3 = \frac{b}{c}\, s$ and $t_3 = \frac{b}{c}\, t$ where $b = \arccos\langle A, C\rangle$.

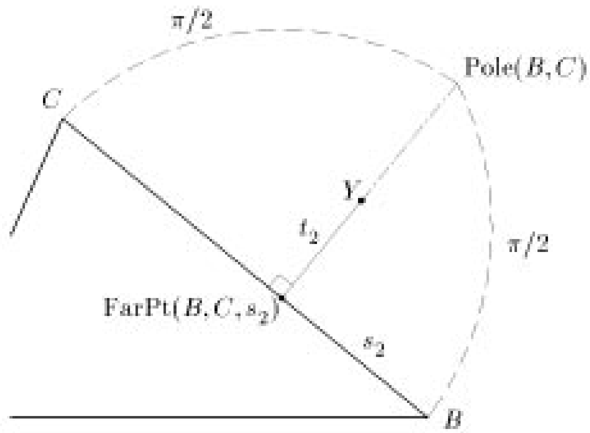

Figure 3. Construction of the point Y on the sphere.

In this case $\triangle XYZ$ is equilateral if and only if:

$$\begin{cases} F(c, b, \alpha, s, t) \coloneqq \langle X, Y - Z\rangle = 0 \\ G(c, b, \alpha, s, t) \coloneqq \langle Y, X - Z\rangle = 0 \end{cases}$$

If $\triangle ABC$ is equilateral, then $a = b = c$, $0 < c < 2\pi/3$, and, by applying spherical trigonometry [**1**, pp. 198–199], we also know that $\alpha = \arccos[\tan(c/2)\, \cot c] = \arccos[(\cos c)/(1 + \cos c)]$. After some manipulation, *Mathematica* indicates that $F(c, c, \alpha, s, t) = 0$ for all values of c, s, t. Because of the rotational symmetry of $\triangle ABC$ about its circumcenter, this result appears to show that $\triangle XYZ$ is always equilateral. Further calculations with *Mathematica* show that the points X, Y, Z are equally spaced on a circle whose center is the circumcenter of $\triangle ABC$.

If $\triangle ABC$ is not equilateral, then what happens is not so clear. We give some numerical results when $c = \pi/4,\ b = \pi/6,\ \alpha = \pi/3$. Figure 4a shows the graphs of F, G for $0 \leq s \leq \pi/4,\ -\pi/2 \leq t \leq \pi/2$ and Figure 4b shows that portion of these two surfaces between the horizontal planes at heights -0.001 and 0. Pairs of s, t that make F and G both zero appear as intersections of surfaces in Figure 4b. Evidently there are two pairs of s, t for which $\triangle XYZ$ is equilateral and *Mathematica* approximates these values to six decimal places as $(s, t) \approx (0.392699, -0.235769)$ and $(s, t) \approx (0.392701, 0.213332)$. The results of letting $-\pi \leq s \leq \pi$ and $-\pi/2 \leq t \leq \pi/2$ are found in Figure 5. Near the center of Figure 5b are found the solution pairs of Figure 4b, but there are also two other solutions near the top and bottom of Figure 5b. These additional solution pairs are approximately $(-1.97296, -0.189836)$ and $(2.90049, 0.349408)$.

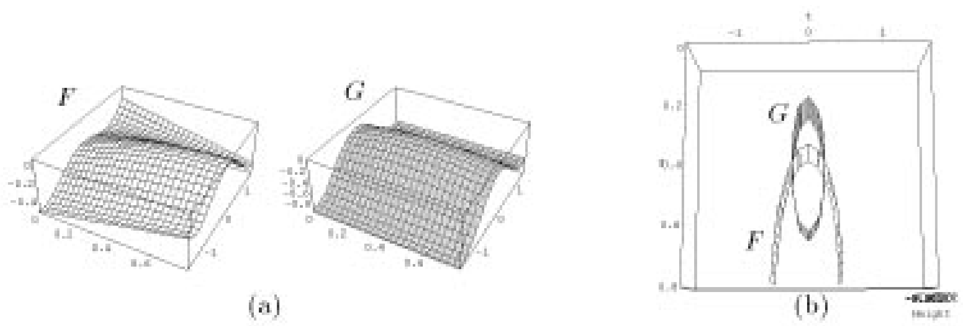

Figure 4. Graphs of F, G for $0 \leq s \leq \pi/4,\ -\pi/2 \leq t \leq \pi/2$.

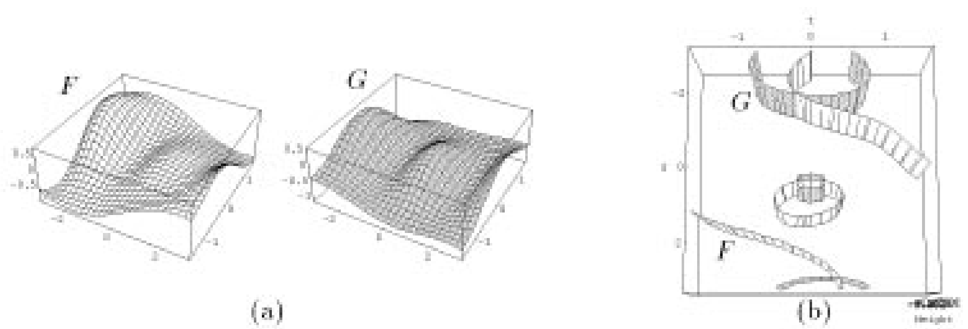

Figure 5. Graphs of F, G for $-\pi \leq s \leq \pi,\ -\pi/2 \leq t \leq \pi/2$.

If we take larger ranges for s and t, even more solution pairs appear. Could it be that there are infinitely many distinct pairs s, t for which $\triangle XYZ$ is equilateral? If so, are there infinitely many distinct, equilateral triangles XYZ? The interested reader is encouraged to investigate these conjectures.

References

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2. J. E. Wetzel, Converses of Napoleon's Theorem, *Amer. Math. Monthly* 99 (1992), 339–351.

3. Wolfram Research Inc., *Mathematica*, Version 2.2.1, Wolfram Research Inc., Champaign, IL, 1992.