

# The Scrambler

or A Family of Vectors at the Amusement Park  
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## *Abstract*

A favorite amusement park ride is used to demonstrate how the vector equation of a complicated curve is derived. The graphing calculator is used to show how the components go together to graph the path. The calculator is used to draw the position, velocity and acceleration vectors. This visualization, which can be easily adapted to other paths, shows the relationships between the three related vectors in an animated way that aids in the understanding of the concepts. Variations of the procedure are discussed.

## *The Scrambler*

One of my favorite rides at the amusement park is the Scrambler. The Scrambler has three long arms which revolve around a central post. At the end of each arm hangs a group of seats on which the victims sit. These seats revolve in a circle at the end of the arm (see Figure 1). The rider travels in a circle which in turn is moving around a larger circle. I've always been intrigued what the path the rider traverses really is — what it would look like from above. And I've wondered how the ride could be made more exciting.

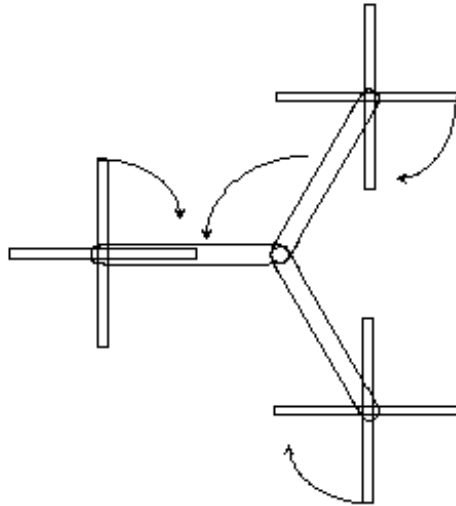


Figure 1

The motion is fairly easy to analyze using vectors. Introduce a coordinate system whose origin is at the center post of the Scrambler. Let  $\vec{OA}$  be the vector from the origin to the end of one of the arms,

and let  $\vec{AB}$  be the vector from the center of that arm to one of the seats as shown in Figure 2. The path is represented by the various positions of the endpoint of  $\vec{OB}$ , since  $\vec{OB} = \vec{OA} + \vec{AB}$

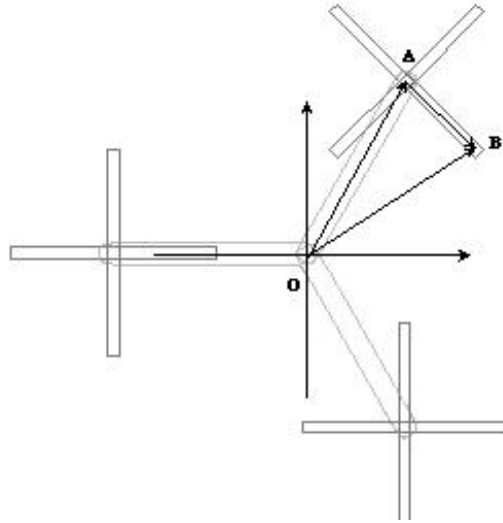


Figure 2

Point A moves in a circle around the origin so it is of the form  $\vec{OA} = (R_1 \cos(t), R_1 \sin(p \cdot t))$ . Where  $R_1$  is the radius of the longer arm and  $t$  is the angle formed with the positive x-axis. Since the Scrambler revolves at a constant rate, we may consider this angle to be our time variable.

Point B moves around a circle with center at A so its equation is of the form  $\vec{AB} = (R_2 \cos(p \cdot t), R_2 \sin(p \cdot t))$ . Here  $R_2$  is the length of the shorter arm.

### **The Parameter p**

Since we will want to consider different relative speeds and directions of rotation of the two arms the parameter  $p$  has been included in the arguments of the second vector.

- The shorter arm will rotate  $|p|$  times for each single revolution of the larger arm.
- Integral values of  $p$  will return the rider to the starting point after one full revolution of the larger arm; non-integral values will not. Most scramblers do not return to the starting point in one revolution.
- If  $p > 0$  both arms will revolve in the same direction (counterclockwise); if  $p < 0$  the arms will rotate in different directions with the shorter arms moving clockwise. Scramblers usually rotate in different directions.

### **The Path**

The path a rider travels is then the sum of these two vectors:

$$\vec{OA} + \vec{AB} = (R_1 \cos(t), R_1 \sin(t)) + (R_2 \cos(p \cdot t), R_2 \sin(p \cdot t)), \text{ or}$$

$$\vec{OB} = (R_1 \cos(t) + R_2 \cos(p \cdot t), R_1 \sin(t) + R_2 \sin(p \cdot t))$$

This equation may be easily graphed with a graphing calculator in parametric mode. Keep  $R_1$  and  $R_2$  constant and start with  $p = -2$ . The path is shown in Figure 3



Figure 3: The motion starts on the right at the point  $(0, R_1 + R_2)$  and moves down and to the left.

### **Seeing the Component Vectors**

By using the programming and drawing features of the graphing calculator one can see the path and how the components go together to make the path. Enter the components  $\vec{OA}$  and  $\vec{AB}$  separately as the first and second functions in parametric mode. Then enter the following program in a graphing calculator (without the line numbers):

1. For n, tmin, tmax, tstep
2. Line 0, 0, xt1(n), yt1(n)
3. Line xt1(n), yt1(n), xt1(n) + xt2(n), yt1(n) + yt2(n)
4. Line 0, 0, xt1(n), yt1(n), 0
5. Line xt1(n), yt1(n), xt1(n) + xt2(n), yt1(n) + yt2(n), 0
6. PtOn xt1(n) + xt2(n), yt1(n) + yt2(n)
7. EndFor

#### **Comment on the program:**

Some adjustments may be necessary depending on the model of the calculator.

**Line 1:** Tying the For Loop variable (n) to the window variable is a convenient way of controlling the program.

**Line 2:** Draws the first vector.

**Line 3:** Draws the second vector attached to the end of the first vector.

**Lines 4 and 5:** Turn the two vectors off for clarity (some graphing calculator do not have this capability. If so, these two lines may be omitted).

Line 6: Turns the point on the curve on.

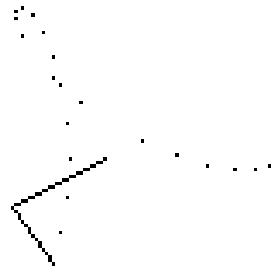


Figure 4: The partially drawn path showing the two component vectors. (The origin is in the center). Note that the points are at equally spaced *time* intervals

### ***Investigating the Path***

As indicated above the value of  $p$  affects the number and kind of loops on the path. Several other paths are shown below. If  $p$  was entered in the parametric equation, the value of  $p$  may be easily changed by storing a new value to  $p$  from the Home Screen.

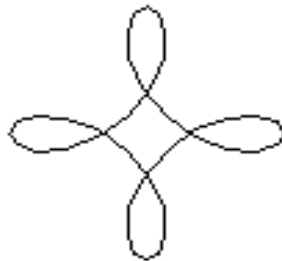


Figure 5:  $p = -3$

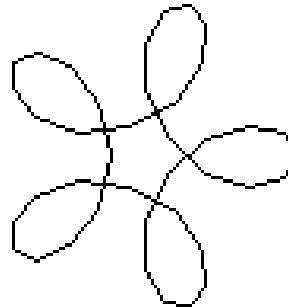


Figure 6:  $p = -4$

There are  $(1 - p)$  loops when the circles revolve in different directions.

Next we look at a typical non-integral value for  $p$ . Now, after one revolution of the large arm, the path does not end at the starting point.

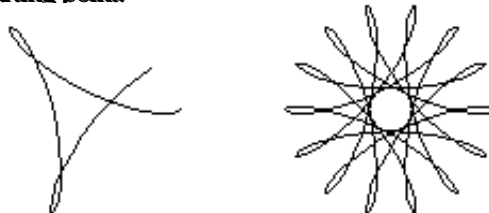


Figure 7: (left)  $p = -1.8$ , 1 revolution and (right) 10 revolutions

Using positive values of  $p$  give very types of paths.

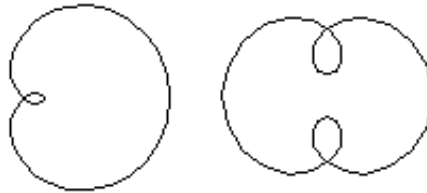


Figure 8:  $p=+2$  Figure 9:  $p=+3$

### ***The Most Exciting Ride — Velocity***

I suppose "the most exciting ride" depends on the rider. From the graphs above it would appear that negative values of  $p$  give more and sharper turns and that the larger  $|p|$  gives more turns. Referring again to Figure 4, the points are spaced at equally spaced *time* intervals. The nearness of a point to the adjacent points is an indication of the speed.

Perhaps a more objective way to study the ride is to consider the velocity. To visualize the velocity it is possible to draw the velocity vectors at any point on the curve. This is done by using the graphing calculator's LINE operation to draw a segment whose length and direction correspond to the velocity vector's coordinates. Put the tail of each vector (the first coordinates in LINE) at the corresponding point on the curve.

Differentiate the position vector to obtain the velocity vector

$$\vec{v} = ((-R_1 \sin(t) - R_2 \cdot p \cdot \sin(p \cdot t)), (R_1 \cos(t) + R_2 \cdot p \cdot \cos(p \cdot t)))$$

Then write a short program to graph the lines and the path. For example enter the path  $\vec{OB}$  as  $xt1(t)$  and  $yt1(t)$  and selected (turned on) and the velocity components as  $xt2(t)$  and  $yt2(t)$  and deselected (turned off) . Run a program with these key lines:

```
For n, tmin, tmax, tstep
Line xt1(n), yt1(n), xt1(n) + xt2(n), yt1(n) + yt2(n)
EndFor
```

The first line is a convenient way to tie the program to the parameter's values. The second line draws a segment from the point on the path in the direction of, and with the length of the velocity vector at that point; that is it draws the velocity vector.



Figure 10: Velocity Vectors At left some are drawn. At right all drawn ( $p = -2$ )

It appears that the longest velocity vector — the fastest part of the ride — occurs in the middle of each section. This is generally true since there must be some slowing down to make the turns.

The same idea may be used to see the acceleration vectors. Enter the acceleration components as  $xt3(t)$  and  $yt3(t)$ . Then enter and run this program which works the same as the last program:

```

For n, tmin, tmax, tstep
Line xt1(n), yt1(n), xt1(n) + xt3(n), yt1(n) + yt3(n)
EndFor

```

The acceleration vectors may be large and not fit well on the screen; some resizing of the window may be required. As you may suspect the acceleration is largest at the ends of the loops when the direction of the velocity is changing the most; it is least when the speed is most. The velocity and acceleration may be compared by graphing the lengths of the velocity and acceleration vectors together as functions of time:

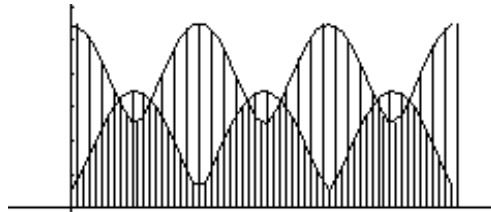


Figure 11: Upper graph is the length of the acceleration vectors vs. time. Lower graph is the speed vs. time.  $0 \leq t \leq 2p$  and  $0 \leq y \leq 60$ .  $p = -2$

### ***Conclusions and Extensions***

The exploration above shows a method for visualizing the velocity and acceleration vectors of a vector equation. The general procedure is to use the LINE operation of a graphing calculator to draw the velocity or acceleration vectors at various points of the curve. The graphs may be produced with a simple program on any graphing calculator with a LINE operation.

- Enter the position equation in the parametric editor as  $xt1(t)$  and  $yt1(t)$ . Leave this on (selected) if you want the path drawn.
- Enter the velocity equations in the parametric editor as  $xt2(t)$  and  $yt2(t)$  and turn them off.
- Enter the acceleration equations in the parametric editor as  $xt3(t)$  and  $yt3(t)$  and turn them off.

With simple variations the entries above may be used to demonstrate

- **The position vectors:** To draw the vectors from the origin to each point on the path use  
**Line 0, 0, xt1(n), yt1(n).**
- **The velocity the vectors:** To attach the velocity vectors to the corresponding points on the path, first draw the path and then draw the velocity vectors with  
**Line xt1(n), yt1(n), xt1(n) + xt2(n), yt1(n) + yt2(n).**
- **The acceleration vectors:** To attach the acceleration vectors to the corresponding points on the path, first draw the path and then draw the acceleration vectors with  
**Line xt1(n), yt1(n), xt1(n) + xt3(n), yt1(n) + yt3(n)**
- **Position and velocity vectors:** Attach the velocity vectors to the ends of the position vectors to demonstrate how the velocity vector “pulls” the position vector to its next position by using a program with these lines  
**Line 0, 0, xt1(n), yt1(n)**  
**Line xt1(n), yt1(n), xt1(n) + xt2(n), yt1(n) + yt2(n)**
- **Velocity and Acceleration vectors:** Attach the acceleration vectors to the ends of the velocity vectors to demonstrate how the acceleration vector “pulls” the velocity vector to its next position by using a program with  
**Line xt1(n), yt1(n), xt1(n) + xt2(n), yt1(n) + yt2(n)**  
**Line xt1(n)+xt2(n), yt1(n) + yt2(n), xt1(n) + xt2(n)+xt3(n), yt1(n)+yt2(n)+ yt3(n)**
- **Translations:** To demonstrate the translation of a figure enter the equation of the figure as xt1(t) and yt1(t) and the translating vector (two constants) as xt2(t) and yt2(t). Then use a program with  
**Line 0, 0, xt1(n), yt1(n)**  
**Line xt1(n), yt1(n), xt1(n) + xt2(n), yt1(n) + yt2(n).**

Interchange the parametric equations (enter the translating vector as xt1(t) and yt1(t) and the equation of the figure as xt2(t) and yt2(t)). Then run the same program.

### **Teaching Suggestion**

- Usually **watching the graph being drawn** is more helpful than seeing the final graph.
- **Simple curves** such as line, circle or parabolas work as well as complicated ones.
- **Circles and ellipses** show nicely that the acceleration vectors always points to the center of these curves. How long are they?
- If you are studying **projectile motion**, Look at the horizontal and vertical components of velocity separately by making first  $yt2(t) = 0$  and then  $xt2(t) = 0$ .

- **Cycloids** of various types make interesting graphs.
- In picking examples be sure to check the graph window values beforehand to find ones that work well.

**Reference**

These curves are called *Trochoids*, *Roulettes*, *Epicycloids* and *Hypocycloids*. (These are overlapping categories.) For more information see *Curves and their Properties* by Robert C. Yates, NCTM, Reston VA, 1952, p. 233 - 236 (Trochoids), p. 175 - 185 (Roulettes) and p. 80 - 85 (Epicycloids and Hypocycloids)