A Model Precalculus Curriculum: The Gateways to Advance Mathematical Thinking Project

Michelle Manes
Education Development Center, Inc.
michelle@edc.org

Education Development Center, Inc. is a private, non-profit education research and development company. EDC does research and development in many areas, including mathematics. In the past, EDC's mathematics work has been primarily at the K-12 level. In fact, in the past 6 years, projects have been funded to develop curricula at the elementary, middle, and high school levels. Recently we have begun doing work at the undergraduate level, and my presented paper is about one of these undergraduate projects: Gateways to Advance Mathematical Thinking.

The GAMT project is funded by the National Science Foundation. Our work is divided into three categories:

- Empirical research in student learning in Linear Algebra
- Curriculum development in precalculus
- Theoretical research on mathematical ways of thinking

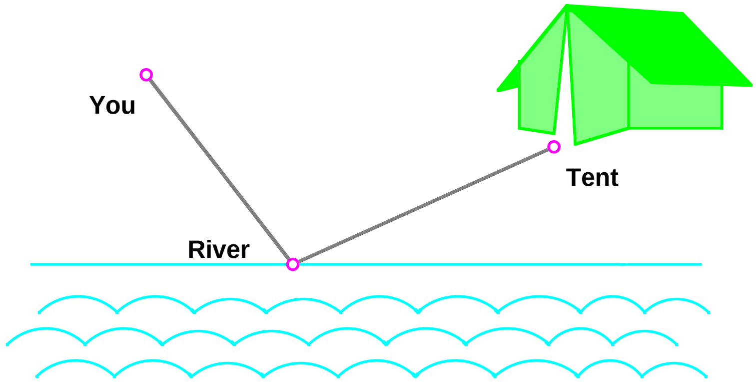
The precalculus development team is working to develop a model for new precalculus curriculum. We will not be developing a full curriculum for publication. Instead, we are outlining the themes around which units might be organized, and then developing one of those units. Some of these themes are

- Change
- Motion
- Limits
- Optimization

We have begun development on the optimization booklet, which is divided into four parts. First, we introduce the notion of optimization, then explore geometric, algebraic, and graphing techniques to solve optimization problems. This paper describes in more detail some of the problems from the geometric techniques section. For algebraic techniques, students can use simple inequalities (like the square of a number is always positive and is zero only when the number is zero, and the algebraic-geometric mean inequality) to solve optimization problems. In the graphing section, students develop graphs to describe optimization problems and they can find the extrema either exactly or approximately, depending on the function they are analyzing. We end with this section because there are clearly many functions that come up for which students cannot find the extrema exactly. This points to the need for a new set of tools, specifically for calculus.

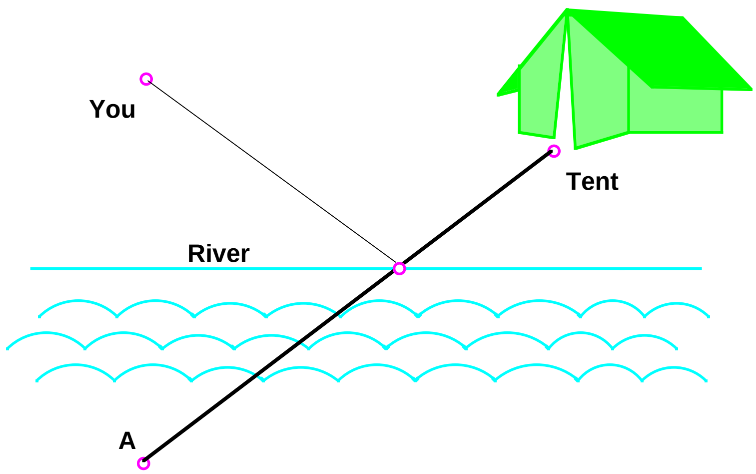
So, here are some problems from the optimization module, with discussion about why we think these are important problems for precalculus students to explore. We start with a fairly well-known problem, but we have some new ideas for solution methods and insights to be gained from it.

> You're on a camping trip. While walking back from a hike, you see that your tent is on fire. Luckily you're holding a bucket and you're near a river. Where should you get the water along the river to minimize your total travel back to the tent? Justify your answer.



Where along the river should you stop?

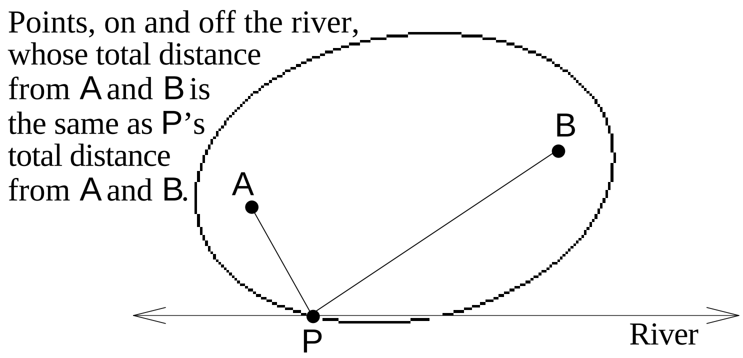
This problem can be solved several ways. The standard solution is to reflect the point labeled "You" over the river to create a new point A. The river is then the perpendicular bisector of the segment between A and "You" so every point on the river is equidistant from these two points. So, we draw the straight segment from A to the Tent, and where that segment crosses the river is the best place to stop and fill the bucket.



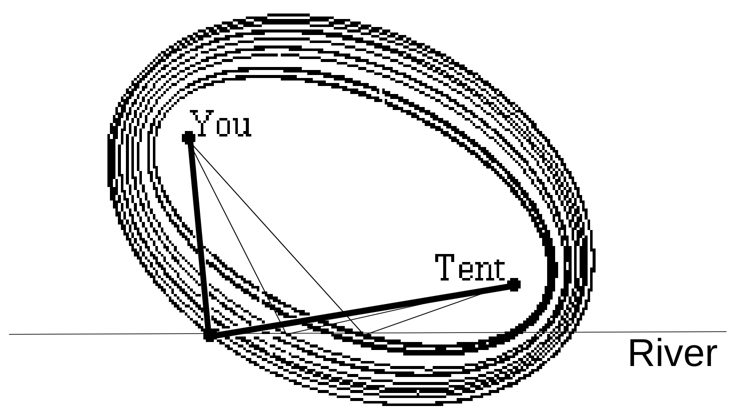
Finding the best place to stop.

There are ways to alter this problem, adding a little difficulty or examination of this technique. For example, suppose you run twice as fast with an empty bucket as with a full bucket. Can you alter this solution to find the new best place to land? Also, this problem can be solved with calculus. I gave it to my husband, who had never seen the usual solution. He put the drawing down on a coordinate plane, wrote out some distance formula, took a derivative, and got an answer. It was a mess, and the answer didn't give him as much insight to the problem situation as this one. But there's still another way to look at it that might buy you even more.

Pick some point on the river, P, and look at *all* the points (not just on the river) whose total distance from you and the tent are the same as P. These points form an ellipse. Most likely, it will look something like this, with the ellipse crossing the river in two places.



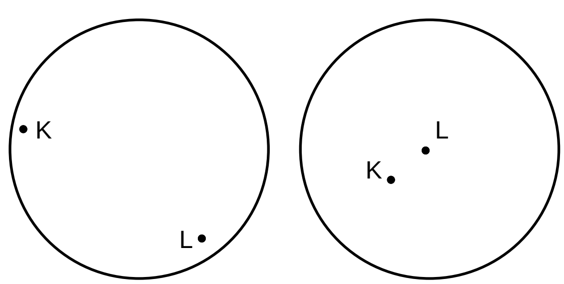
The points on the river but inside the ellipse have a smaller total distance from You and the Tent, so you can picture sliding P over to these points, and looking at the new ellipse that forms.



You can see this whole family of ellipses, and the one that's tangent to the river will have the smallest total distance; in other words, it's the place to stop and fill your bucket. Any smaller ellipse won't cross the river at all, and any bigger one will cross it in two points, leaving some better places inside.

So why is this a nice way to look at the problem? Well, besides being a nice way to connect to conics and to contour lines, consider the following problem.

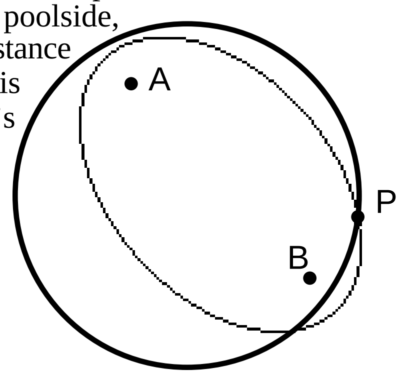
> Natasha is in a circular swimming pool at K. She wants to swim to L but first she wants to swim to the edge of the pool to leave her sunglasses. Explain how to find the best place to put the sunglasses to minimize the total amount of swimming.



Two possible arrangements for this problem

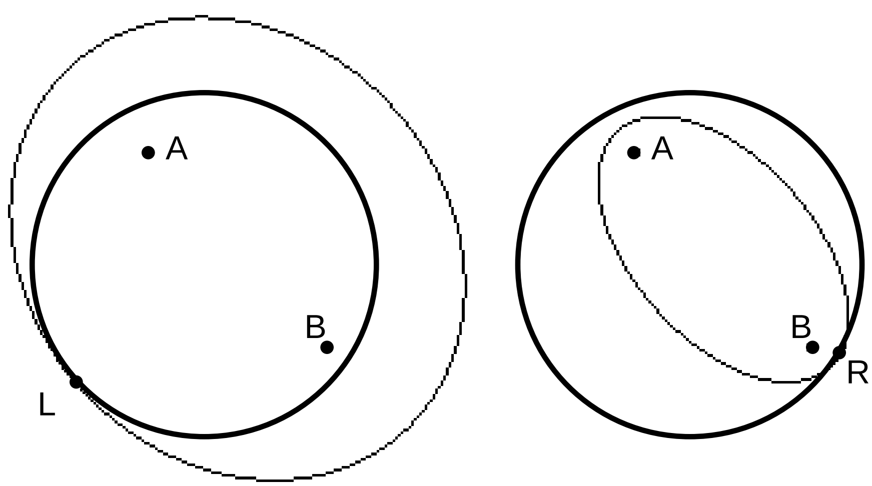
The connection to the previous problem is clear: you want to minimize the total travel between two points with a required stop in between. The reflection technique, however, won't work here because you have a circle instead of a nice, straight river. And the calculus solution gets even more messy. But think about the ellipses again.

Choose P somewhere on the edge of the pool, and consider all the points the same total distance from K and L as P. Again, most likely it crosses the circular pool in at least two places. Any points on the edge of the pool, but inside the ellipse, are better choices of P, so slide P over, make a new ellipse, and keep going until you get a tangent ellipse.

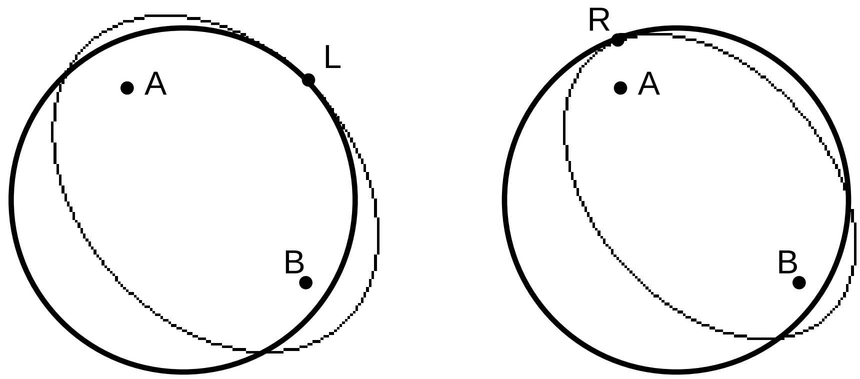


The ellipse shows all points, on and off the poolside, whose total distance from A and B is the same as P's total distance from A and B.

In this situation, however, there's more than one possible tangency. The one we want—the absolute minimum—is an ellipse that is completely inside the pool, except for one or two internal tangencies. We might also have an ellipse that completely contains the pool, and is tangent on the outside. This would be an absolute maximum.



You can even imagine finding relative maxima and minima using this technique.



This exposes students to many of the ideas in calculus, using simple geometric techniques. Our goal for the entire project is to have students do these kinds of problems—interesting and accessible problems that point towards calculus.